Lecture 01 Mathematics

The content in Lecture 01 can be partially found in **Appendix B in Serway/Jewett's** textbook of "Physics for Scientists and Engineers with Modern Physics".

In this lecture, we will review background knowledge of mathematics used in physics. We help to review what you learned in senior high school and introduce you new, applied mathematics of differentiation, integration, Taylor's expansion, a simple 1st order differential equation, and a 2nd order differential equation.

1.1 Quadratic Equation

Sometimes you may use a variable and find a relation forming a quadratic equation for solving the variable. For example, you assume the distance be the variable x and you find the relation of $bx = -c - ax^2$. Thus, you get a quadratic equation

$$ax^2 + bx + c = 0.$$

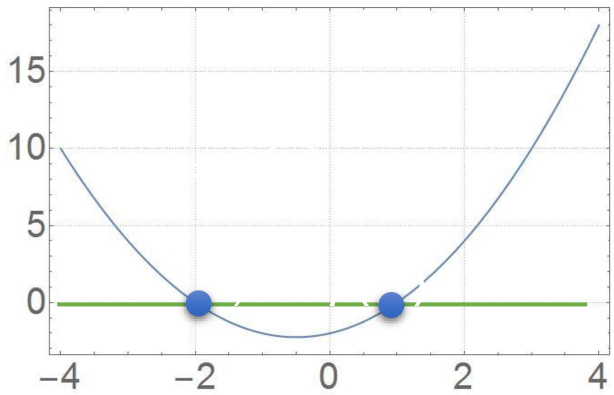
Solve it, you may go through the step

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}.$$

Finally, you obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The following graph gives you a concept of solving the equation of $x^2 + x - 2 = 0$. The solutions tell us the interception points of the two functions, $y(x) = f(x) = 0$ and $y(x) = f(x) = x^2 + x - 2$.



1.2 Linear Equations

If you have more than one variables, you need to have the same number of equations to find the solution of variables. For example, we have two variables x and y, satisfying the following two equations.

$$y - x = 2$$
$$3x - 2y = 8$$

We first put the equations into standard form

$$x - y = -2$$
$$3x - 2y = 8,$$

and use the Cramer's rule (formula of determinants) to find values for the two variables.

$$x = \frac{\begin{vmatrix} -2 & -1 \\ 8 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}} = \frac{12}{1} = 12$$

$$y = \frac{\begin{vmatrix} 1 & -2 \\ 3 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}} = \frac{14}{1} = 14$$

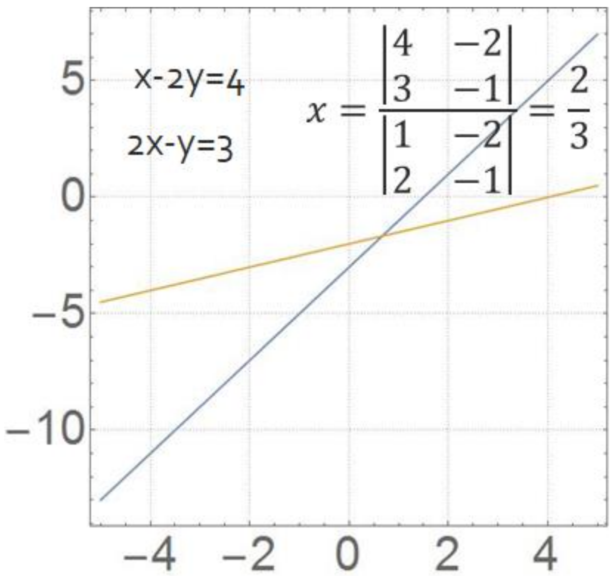
Use determinant formula to sole a system of equations for three variables x, y and z:

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

The determinant formula gives you the solutions such as

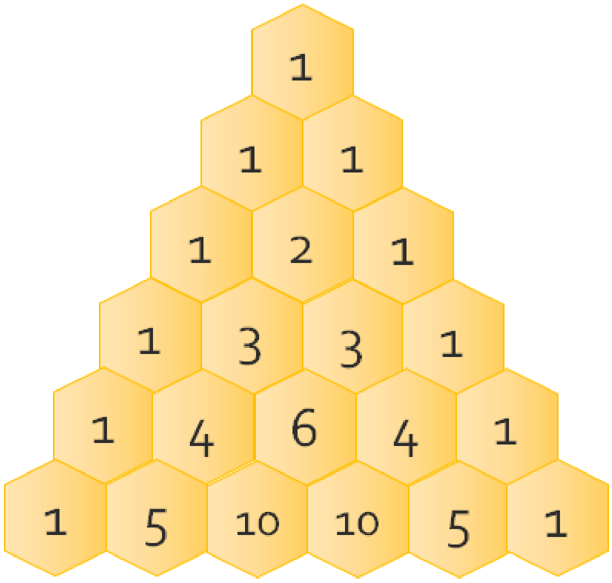
$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

The following graph draws lines of two equations with an interception point of a solution given by the determinant formula.



1.3 Binomial Series

The following figure shows Pascal's triangle which gives you coefficients in the expansion of $(x+1)^n$.



$$(x+1)^2 = (x+1)(x+1) = x^2 + 2x + 1 = C_2^2 x^2 + C_1^2 x^1 + C_0^2 x^0$$

$$(x+1)^n = C_n^n x^n + C_{n-1}^n x^{n-1} + C_{n-2}^n x^{n-2} + \cdots$$

$$(x+1)^n = x^n + n x^{n-1} + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

1.4 Power & Exponent

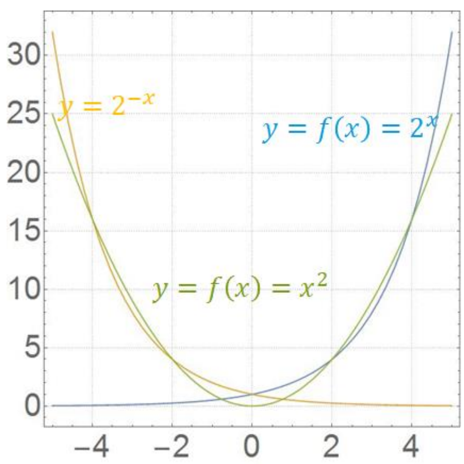
$$2^m \cdot 2^n = 2^{m+n}$$

$$2^0 = x^0 = 1$$

$$x^m = x^{m-n} \cdot x^n$$

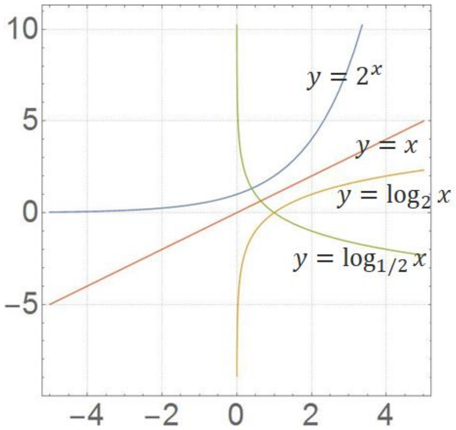
$$m = 0 \rightarrow 1 = x^{-n} x^n \rightarrow x^{-n} = \frac{1}{x^n}$$

$$A \cdot A = x = x^{\frac{1}{2}} x^{\frac{1}{2}} \rightarrow A = x^{\frac{1}{2}} = \sqrt{x}$$

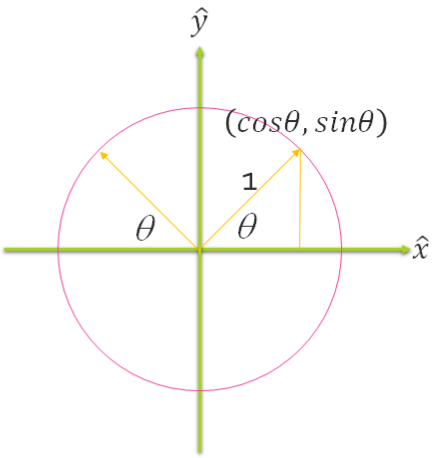


1.5 Logarithms

$$y = a^x$$
$$x = log_a(y)$$
$$100 = 10^2$$
$$log_{10}(100) = 2$$
$$log_{10}(1) = 0$$
$$y_1 = a^x, y_2 = a^y$$
$$y_1 y_2 = a^x a^y = a^{x+y}$$
$$\log_a(y_1 y_2) = x + y = \log_a(y_1) + \log_a(y_2)$$
$$e = 2.718281828459045$$



1.6 Trigonal Functions



Trigonometric Identities

$$sin(A \pm B)$$
$$= sinAcosB \pm cosAsinB$$
$$\cos(A \pm B)$$
$$= cosAcosB \mp sinAsinB$$
$$sin(\pi - \theta) = sin\theta$$
$$cos(\pi - \theta) = -cos\theta$$

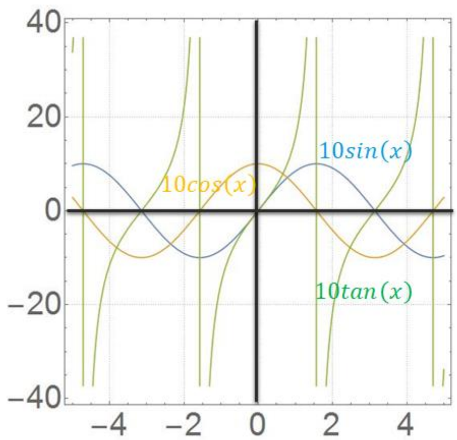
$$sin\left(\frac{\pi}{2}-\theta\right) = cos\theta$$

$$sin(A) = sin\left(\frac{A+B}{2}+\frac{A-B}{2}\right)$$

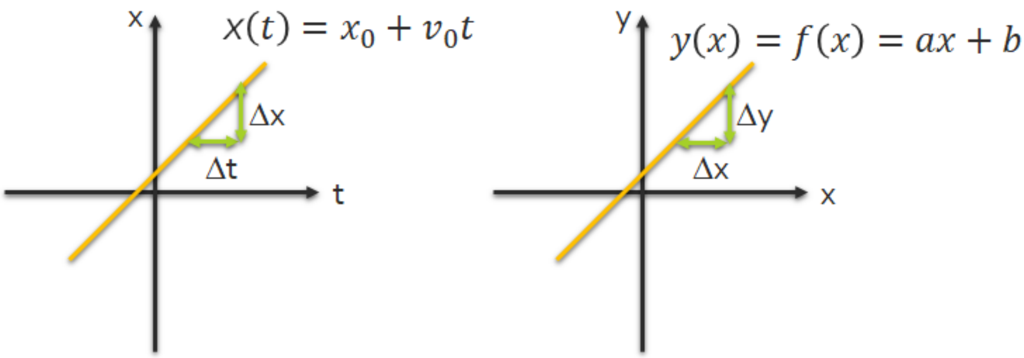
$$sin(A) = sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)+cos\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)$$

$$sin(B) = sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)-cos\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)$$

$$sin(A)+sin(B) = 2sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$



1.7 Differential Calculus



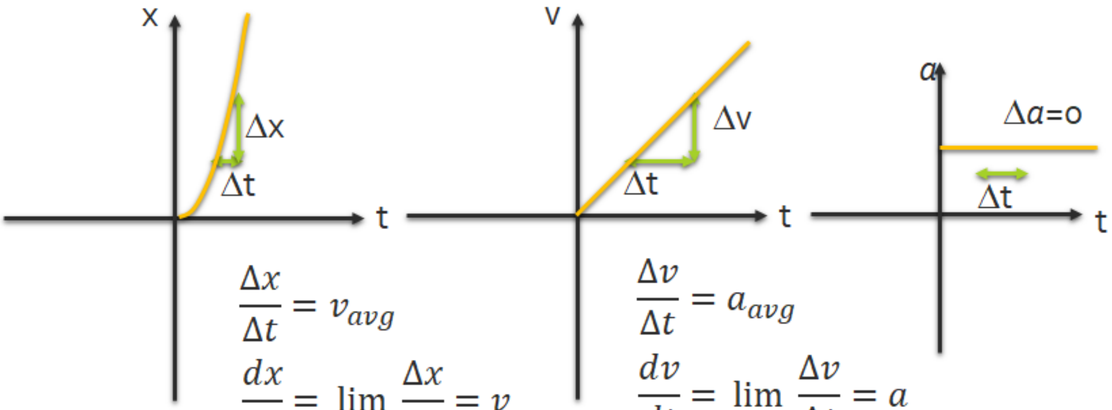
Differential and integral calculation are especially designed for expressing particles in motion. For example, a constant speed motion in one-dimensional space is described by the function of position X as a function of time t, referring to the left panel in the figure above. The constant velocity of the motion can be derived as

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = v_0.$$

The calculation is similar to finding the slope of a line or curve as shown in the right panel of the figure shown above and the slope can be expressed as

$$m = \frac{\Delta y}{\Delta x} = \frac{y(x_2) - y(x_1)}{x_2 - x_1}.$$

On the other hand, a particle under a constant acceleration motion gives you similar concepts.



$$\frac{\Delta x}{\Delta t} = v_{avg} \qquad \frac{\Delta v}{\Delta t} = a_{avg}$$

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = v \qquad \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = a$$

The differentiation is just the way to calculate the slope between two points on the curve while moving the two points approaching as closely as possible. Thus, the slope of the two points turns out to be the slope of a tangent line at the two approaching points.

The definition of the differential calculation of a function is given as

$$\frac{df(x)}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Let's use this definition to find out slopes of power functions. The first kind of basic functions is the power function $f(x) = x^n$.

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^n + C_1^n x^{n-1}(\Delta x) + C_2^n x^{n-2}(\Delta x)^2 + \cdots - x^n}{\Delta x}$$

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} (nx^{n-1} + C_2^n x^{n-2}(\Delta x) + \cdots) = nx^{n-1}$$

If we know that for very small θ, the sine and cosine functions of θ have the relations of $\sin(\theta) \cong \theta$ and $\cos(\theta) \cong 1$. We can then derive the slope of sine and cosine functions. They are the second kind of basic functions.

$$\frac{d}{dx}(\sin(x)) = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x) \cdot 1 + \cos(x) \cdot (\Delta x) - \sin(x)}{\Delta x} = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos(x)\cos(\Delta x) - \sin(x)\sin(\Delta x) - \cos(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos(x) \cdot 1 - \sin(x) \cdot (\Delta x) - \cos(x)}{\Delta x} = -\sin(x)$$

The product and quotient rules are commonly used in differentiation. They are expressed as follows.

$$\frac{d}{dx}\big(f(x)g(x)\big) = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\big(g(x)\big)^{-1}\right)$$

$$= \frac{df(x)}{dx}\big(g(x)\big)^{-1} + f(x)\left(-\big(g(x)\big)^{-2}\frac{dg(x)}{dx}\right)$$

$$= \frac{\frac{df(x)}{dx}g(x) - f(x)\frac{dg(x)}{dx}}{\big(g(x)\big)^2}$$

We can use the quotient rule to derive the differential of $\tan(x)$.

$$\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{\cos^2(x)} = \sec^2(x)$$

The last rule that is commonly used in calculus is the chain rule.

$$\frac{df\big(g(x)\big)}{dx} = \frac{df(g)}{dg}\frac{dg(x)}{dx}$$

1.8 Partial Differential

For a function of two more variables, we can either do complete or partial differential calculation. The complete and partial differential of a function $f(x,y)$ are denoted as $\frac{df(x,y)}{dx}$ and $\frac{\partial f(x,y)}{\partial x}$.

For the partial differentiation, we take other variables as constants. It means that the variables, like x and y, are independent. For the complete differentiation, we need to consider the variables have functional dependences such as $y = y(x)$.

We learn functional dependences for functions. When we use coordinate system, we prefer to use orthogonal coordinates. That means the variables x, y and z are independent. In such an orthogonal and independent variable system, we can easily treat differentiation and integration.

We illustrate by using a function $f(x,y) = x^2 + 2xy + y^2$. The complete differentiation gives the result of

$$\frac{df(x,y)}{dx} = 2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = (2x + 2y) + (2x + 2y)\frac{dy}{dx}$$

$$\frac{\partial f(x,y)}{\partial x} = 2x + 2y$$

$$\frac{\partial f(x,y)}{\partial y} = 2x + 2y$$

$$\frac{df(x,y)}{dx} = \frac{\partial f(x,y)}{\partial x}\frac{dx}{dx} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx}$$

$$\frac{df(x,y)}{dx} = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx}$$

The idea can be generalized for three variable functions.

$$\frac{df(x,y)}{dx} = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx} + \frac{\partial f(x,y)}{\partial z}\frac{dz}{dx}$$

1.9 Taylor Expansion

After we learned the differentiation, we can use it as a first order, linear approximation.

$$f(x) \cong f(x_0) + \frac{df(x)}{dx}_{x=x_0}(x - x_0)$$

The idea is the same as finding the coefficients c_0 and c_1 of the linear approximation equation

$$f(x) \cong c_0(x - x_0)^0 + c_1(x - x_0).$$

The Taylor expansion is just a higher order approximation of the polynomial function.

$$f(x) \cong c_0(x - x_0)^0 + c_1(x - x_0)^1 + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \cdots$$

If x is very close to x_0, we cannot tell any difference between the real $f(x)$ and the Taylor expansion. The coefficients in Taylor expansion are just higher order differentials.

$$c_0 = f(x)_{x=x_0}$$

$$c_1 = \frac{1}{1!}\frac{df(x)}{dx}_{x=x_0}$$

$$c_2 = \frac{1}{2!}\frac{d^2f(x)}{dx^2}_{x=x_0}$$

$$c_2 = \frac{1}{3!}\frac{d^3f(x)}{dx^3}_{x=x_0}$$

The factorial $n!$ is related to the differential of the corresponding power function. If the original function is polynomial, we can use Taylor expansion to find exactly the same original function. For example, the function $f(x) = x^2 + 2x$ is a polynomial function and its derivatives are $\frac{df}{dx} = 2x + 2$, $\frac{d^2 f}{dx^2} = 2$, and $\frac{d^n f}{dx^n}_{n\geq 3} = 0$. We can find the original function using Taylor expansion approximation if we know values of the function and all derivatives of the function. For example, when we know $f(3) = 15$, $f'(3) = 8$, $f^{(2)}(3) = 2$, and $f^{(n)}(3)_{n\geq 3} = 0$, we can use Taylor's expansion to obtain

$$f(x) \cong f(x_0) + \frac{1}{1!}\frac{df(x)}{dx}\bigg|_{x=x_0}(x - x_0) + \frac{1}{2!}\frac{d^2 f(x)}{dx^2}\bigg|_{x=x_0}(x - x_0)^2 + \cdots.$$

$$f(x) = 15 + 8(x - 3) + \frac{2}{2!}(x - 3)^2 + 0$$

$$f(x) = x^2 + 2x$$

The function obtained from Taylor expansion is exactly the same as the original function since it's a polynomial function. The most useful Taylor expansion equation is the expansion at $x = 0$, given as follows.

$$f(x) \cong f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \cdots.$$

When you learn Calculus, you will know the original definition of the exponential function

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \cong 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

It is similar to the Taylor expansion result of the original function. With this definition, we can calculate the derivatives of the third kind of basic functions.

$$\frac{d}{dx}e^x \cong \left(\frac{d}{dx}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x$$

The fourth kind of basic function is the logarithm function $\log_e(x) = \ln(x)$. The philosophy is that we cannot differentiate the unknown function from unknown calculations. We shall use chain rule and the know derivative calculation so we change variable from x to y.

$$y = \ln(x) \rightarrow x = e^y$$

$$\left(\frac{d}{dx}\right)x = \left(\frac{d}{dx}\right)e^y$$

Again, we cannot derive $\left(\frac{d}{dx}\right)e^y$ so we turn to use chain rule.

$$\left(\frac{d}{dx}\right)x = \left(\frac{d}{dy}\right)(e^y)\,\frac{dy}{dx}$$

$$1 = e^y \frac{dy}{dx} \rightarrow \frac{dy}{dx} = e^{-y} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{d\ln(x)}{dx} = \frac{1}{x}$$

We can also derive Taylor expansion of sinusoidal functions.

$$\sin(x) \cong \sin(0) + \frac{\sin'(0)}{1!}x + \frac{\sin''(0)}{2!}x^2 + \frac{\sin'''(0)}{3!}x^3 + \cdots$$

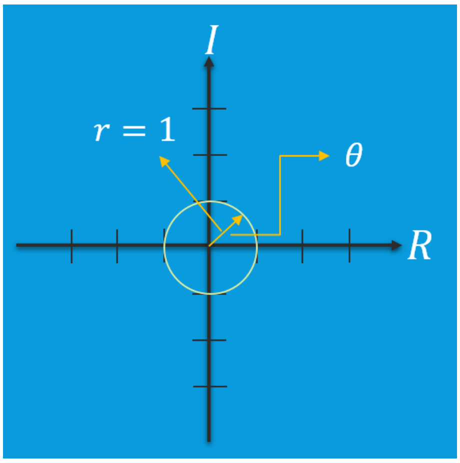
$$\sin(x) \cong x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots$$

$$\cos(x) \cong \cos(0) + \frac{\cos'(0)}{1!}x + \frac{\cos''(0)}{2!}x^2 + \frac{\cos'''(0)}{3!}x^3 + \cdots$$

$$\cos(x) \cong 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots$$

1.10 Complex Numbers

We usually met the problem when we solve the equations $x^2 = -1$ and $x^2 = -n$. The numbers of $\pm\sqrt{-1}$ seems not interactive with normal real numbers. We propose an independent relation between pure imaginary numbers (like $\sqrt{-1}$) and real numbers. The easiest independent variables are x and y variables on the xy coordinate system. Here R and I are real numbers and pure imaginary numbers, respectively. The combination of real and imaginary numbers give you complex numbers.



We usually denote the unit imaginary number $\sqrt{-1}$ as i.

$$i = \sqrt{-1} \rightarrow i^2 = -1, i^3 = -i, i^4 = 1, \ldots$$

$$\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1} = \sqrt{5}i$$

Now we introduce a notation e^{ix} to denote the unit complex number.

$$e^{ix} = 1 + \frac{(ix)}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

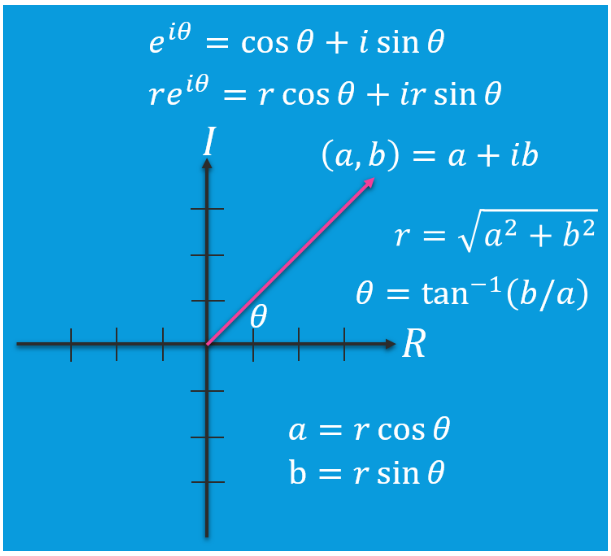
$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots\right) + i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots\right)$$

$$e^{ix} = \cos(x) + i\sin(x)$$

We usually plot a circle of unit radius on the plane of complex numbers and replace x by θ shown in the figure above. It is clear that the real part of $e^{i\theta}$ is like the projection component of $\cos(\theta)$ on the axis of real numbers.

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

We also use this notation to change the normal expression of a complex number $a + ib$, where a and b are two real numbers, to the notation of $r\,e^{i\theta}$.



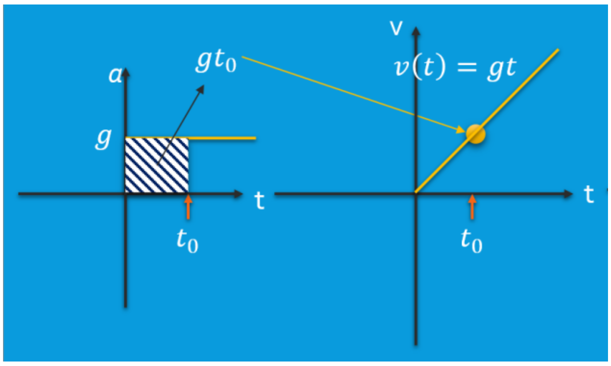
We can use the notation $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to derive again the addition rules for sinusoidal functions.

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$
$$= e^{i\alpha} \cdot e^{i\beta} = (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta))$$
$$= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta))$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

1.11 Integration

The development of calculus (differentiation and integration) is directly related to Newton's classical mechanics. Let's start from the physics problem of an free falling object. For a free falling object, the acceleration is a constant g. We define the positive direction of the motion as the direction pointing to the center of the Earth. If we know the initial position and velocity of an object, we can find its following motion.

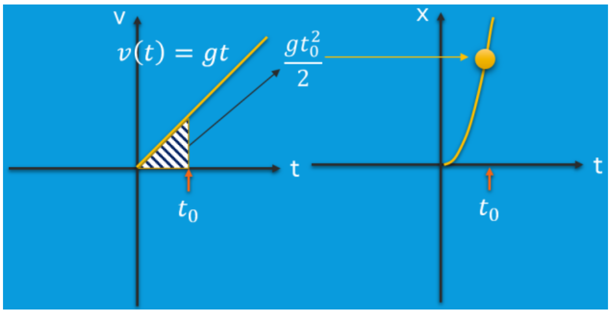


We know that the differential of velocity by time is the acceleration, denoted as $\frac{dv(t)}{dt} = a(t)$. For a free falling object, the acceleration is a constant and it is expressed as $a(t) = g$. In the figure above, we show $a(t)$ ($a - t$ diagram) on the left panel and $v(t)$ ($v - t$ diagram) on the right panel.

$$\frac{dv(t)}{dt} = a(t) \rightarrow d\big(v(t)\big) = a(t)dt = gdt$$

The integration on the $a - t$ diagram from $t = 0$ to $t = t_0$ is just to estimate the area (gt_0) under the curve. The area of the area on the $a - t$ diagram gives you the difference of function values $v(t_0) - v(0)$. The calculation procedure can be expressed by the integration shown here.

$$\int_{v(0)=0}^{v(t_0)} dv = \int_0^{t_0} gdt$$
$$v(t_0) - 0 = gt_0 - 0$$
$$v(t_0) = gt_0$$
$$v(t) = gt$$



The differential of position by time gives the velocity, denoted as

$$\frac{dx(t)}{dt} = v(t) = gt.$$

The integration on $g - t$ diagram is to estimate the area under the line $v(t) = gt$.

The area between $t = 0$ and $t = t_0$ is $\frac{1}{2}(t_0)(gt_0) = \frac{gt_0^2}{2}$. The procedure of calculation is shown here.

$$dx = gtdt$$

$$\int_{x(0)=0}^{x(t_0)} dx = \int_0^{t_0} gtdt$$

$$[x]_{x=0}^{x=x(t_0)} = \left[\frac{gt^2}{2}\right]_{t=0}^{t=t_0} \to x(t_0) = \frac{1}{2}gt_0^2$$

$$x(t) = \frac{1}{2}gt^2$$

The important concept learned from the diagram is that a small variation of function $d(f(x))$ is equal to multiplication of the slope function $\frac{d(f(x))}{dx}$ and the small variation dx.

$$df(x) = \frac{df(x)}{dx}dx$$

The idea can be used to understand the chain rule.

$$\frac{df(g(x))}{dx} = \frac{df(g)}{dg}\frac{dg(x)}{dx} \to df(g(x)) = \frac{df(g)}{dg}\frac{dg(x)}{dx}dx$$

The variation of g is equal to the multiplication of the slope $\frac{dg(x)}{dx}$ and the variation of x. Then, the variation of f is equal to the multiplication of the slope $\frac{df(g)}{dg}$ and the variation of g.

Here we remind you the differential function of some basic functions.

$$f_1(x) = x^n \to \frac{df_1(x)}{dx} = nx^{n-1}$$

$$f_2(x) = \sin(x) \to \frac{df_2(x)}{dx} = \cos(x)$$

$$f_3(x) = e^x \to \frac{df_3(x)}{dx} = e^x$$

$$f_4(x) = \ln(x) \to \frac{df_4(x)}{dx} = \frac{1}{x}$$

From the above differential calculation, we can find the variation of functions as follows.

$$d(f_1(x)) = (nx^{n-1})dx \to d(x^n) = (nx^{n-1})dx$$

$$d(f_2(x)) = (\cos(x))dx \to d(\sin(x)) = (\cos(x))dx$$

$$d(f_3(x)) = (e^x)dx \to d(e^x) = (e^x)dx$$

$$d(f_4(x)) = \left(\frac{1}{x}\right)dx \to d(\ln(x)) = \left(\frac{1}{x}\right)dx$$

The variations of functions give you integration results.

$$\int_0^{x'} nx^{n-1}dx = \int_0^{x'} d(x^n) = [x^n]_{x=0}^{x=x'} = x'^n$$

$$\int_0^{x'} x^n dx = \int_0^{x'} d\left(\frac{x^{n+1}}{n+1}\right) = \left[\frac{x^{n+1}}{n+1}\right]_{x=0}^{x=x'} = \frac{x'^{n+1}}{n+1}$$

$$\int_0^{x'} \cos(x)\, dx = \int_0^{x'} d(\sin(x)) = \sin(x')$$

$$\int_0^{x'} e^x dx = \int_0^{x'} d(e^x) = e^{x'} - e^0$$

$$\int_1^{x'} \frac{1}{x} dx = \int_1^{x'} d(\ln(x)) = \ln(x') - \ln(1)$$

The method of variations of functions can be used to solve more difficult calculations and it is slightly different from the method of changes of variables.

$$\int_0^{x'} \frac{2x}{x^2+1} dx = \int_0^{x'} \frac{1}{x^2+1} d(x^2) = \int_0^{x'} \frac{1}{x^2+1} d(x^2+1) = \int_0^{x'} d(\ln(x^2+1))$$
$$= \ln\left(x'^2+1\right) - \ln(1)$$

1.12 1st Order Differential Equation

Physicists like to observe phenomena and find a mathematical expression to describe the phenomena. For example, it is observed that the growth rate of population is proportional to the population at that time. We express the population at time t as $P(t)$ thus the growth rate of population is $\frac{dP(t)}{dt}$. The observed phenomena just gives you an equation.

$$\frac{dP(t)}{dt} \propto P(t) \rightarrow \frac{dP}{dt} = kP$$

The equation is the 1st order differential equation. It is usually accompanied by an initial condition of $P(t_0) = P_0$ at $t = t_0$.

To solve the 1st order differential equation, we can either try a solution or use the separation of variables. Here we show you the procedure of the separation of variables. Note that we have two variables P and t here. Try to put P on the left and t on the right of the equation. Then, integrate both sides to get the solution.

$$dP(t) = kP(t)dt$$

$$\frac{dP(t)}{P(t)} = kdt$$

$$\frac{dP}{P} = kdt$$

$$\frac{dP}{P} = kdt$$

$$\int_{P_0}^{P(t)} \frac{dP}{P} = \int_{t_0}^{t} kdt$$

$$\int_{P_0}^{P(t)} d(\ln(P)) = k\int_{t_0}^{t} dt$$

$$[\ln(P)]_{P=P_0}^{P=P(t)} = k[t]_{t=t_0}^{t=t}$$

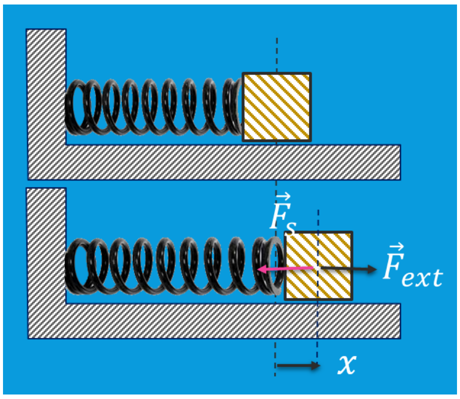
$$\ln\big(P(t)\big) - \ln(P_0) = k(t - t_0)$$

$$\ln\left(\frac{P(t)}{P_0}\right) = k(t - t_0)$$

$$\frac{P(t)}{P_0} = e^{k(t-t_0)}$$

$$P(t) = P_0 e^{k(t-t_0)}$$

1.13 2nd Order Differential Equation



When we study the oscillatory motion in Chapter 15, we will encounter the problem of solving the 2nd order differential equation. Let's take a look at the phenomena. Assume a block of mass m is attached on one end of a spring and the other end of the spring is fixed on the wall. When we pull the block a distance x from its equilibrium position, the block will be exerted by a force $-kx$. We can write the force equation a and the resulting acceleration as

$$F = -kx = ma.$$

We know that the acceleration is dependent on the displacement and time as

$$a = \frac{dv}{dt} = \left(\frac{d}{dt}\right)v = \left(\frac{d}{dt}\right)\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$

The 2nd order differential equation is obtained.

$$-kx = m\frac{d^2x}{dt^2} \rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

As a second step, we need to find solutions of the 2nd order differential equation $m\frac{d^2x}{dt^2} + kx = 0$. To solve it, we need to try a solution $x(t) = A\sin(Bt)$.

Put the trial solution into the differential equation to find any regulations for either A or B.

$$\frac{dx}{dt} = \left(\frac{d}{dt}\right)(A\sin(Bt)) = A\left(\frac{d(\sin(Bt))}{d(Bt)}\right)\left(\frac{d(Bt)}{dt}\right)$$

$$= A\cos(Bt)\,B = AB\cos(Bt)$$

$$\frac{d^2x}{dt^2} = AB\frac{d\cos(Bt)}{d(Bt)}\frac{d(Bt)}{dt} = -AB^2\sin(Bt)$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\rightarrow m(-AB^2\sin(Bt)) + kA\sin(Bt) = 0$$

$$A\sin(Bt)\,(k - mB^2) = 0$$

$$k - mB^2 = 0 \rightarrow B = \pm\sqrt{\frac{k}{m}}$$

Here we choose $B = \sqrt{\frac{k}{m}}$ and the solution is $x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right)$. You can follow the same sequence to confirm that $x(t) = C\cos\left(\sqrt{\frac{k}{m}}t\right)$ is another solution. The linear combination of both solutions are also a solution of the differential equation. The complete solution is given here.

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)$$

The solution can be rewritten as

$$x(t) = R\sin\left(\sqrt{\frac{k}{m}}t + \theta\right),$$

where $R = \sqrt{A^2 + B^2}$ and $\tan(\theta) = \frac{B}{A}$.

Exercise:

1. Please draw the function $x^2 + \left(y - x^{\frac{2}{3}}\right)^2 = 1$ on the xy plane.
2. Please use Cramer's rule to solve the system of differential equations: $2x - y = 5, x + 3y = -1$. $(x = 2, y = -1)$

3. Please use Cramer's rule to solve the system of differential equations: $x + y - z = 1, x + y + z = 2, x - y = 3$. ($x = \frac{9}{4}, y = -\frac{3}{4}, z = \frac{1}{2}$)

4. Please expand the term of $(x + \Delta x)^4$. ($x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4$)

5. Use the relation of $\frac{d(e^x)}{dx} = e^x$ and change of variables $y = e^x \rightarrow x = \ln(y)$ to derive the equation $\frac{d(\ln(y))}{dy} = \frac{1}{y}$.

6. Please change the complex number $a + ib$ to $re^{i\theta}$. Use a, b to express r, θ. ($r = \sqrt{a^2 + b^2}, \theta = \tan^{-1}(\theta)$)

7. Please use the relation for complex numbers of $e^{i\theta} = \cos\theta + i\sin\theta$ to derive the expression of $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$. ($e^{i2\theta} = e^{i\theta}e^{i\theta} \rightarrow \cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2$)

8. Please use the differential operation of $\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ to find the total differential of x^3 that is to evaluate $\frac{d(x^3)}{dx}$.

9. Please calculate the derivative of the function $2x^3 + 4x^2 - x - 5$ with respect to the variable x. ($6x^2 + 8x - 1$)

10. Please calculate the second derivative: $\frac{d^2}{dx^2}\left(2x^3 + 3 + 5\ln(x) + \frac{4}{x^2}\right)$. ($12x - \frac{5}{x^2} + \frac{24}{x^4}$)

11. Please use the differential operation of $\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ to find the total differential of $\sin(x)$ that is to evaluate $\frac{d(\sin(x))}{dx}$. (Please note that for an infinitesimal Δx, $\cos(\Delta x) \cong 1$ and $\sin(\Delta x) \cong \Delta x$)

12. Please use the relations $\frac{d(\sin(x))}{dx} = \cos(x)$, $\frac{d(\cos(x))}{dx} = -\sin(x)$ and the multiplication rules for differential operation to calculate $\frac{d(\tan(x))}{dx}$ and $\frac{d(\sec(x))}{dx}$. ($\sec^2 x, \sec x \tan x$)

13. Please use complete differential calculation to calculate $\frac{d}{dx}(\sin(\cos(x)))$, $\frac{d}{dx}(x^2 + \tan(y(x)))$. ($-\cos(\cos(x))\sin(x), 2x + \sec^2(y)\frac{dy}{dx}$)

14. Please use Taylor expansion to find the first three non-zero terms near $x=0$ for the function $f(x)=\sin(x)$. ($\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}$)

15. For a given function $f(x)=x^2+3x+7$, please calculate $f(1), f'(1), f''(1), f'''(1)$. Please use Taylor expansion and the values of $f(1), f'(1), f''(1), f'''(1)$ at $x=1$ to find the original function $f(x)$ back.

16. Please calculate the partial differential of $\frac{\partial}{\partial x}(x^2 z+xyz+y^2+8z)$. ($2xz+yz$)

17. Please use partial differential calculation to calculate $\frac{\partial}{\partial x}(x^2+\tan(y))$, $\frac{\partial}{\partial y}(x^2+\tan(y))$. ($2x, \sec^2(y)$)

18. Please carry out the integration of $\int_1^2(\sqrt{x}+2x^2)\,dx$. ($4+\frac{4\sqrt{2}}{3}$)

19. Please calculate the integration $\int_0^{2\pi}\cos(2x)\,dx$ and $\int_0^{2\pi}\sin^2(x)\,dx$. ($0, \pi$)

20. Please calculate the integration of $\int_0^{\frac{\pi}{4}}(\sec^2(x)+\cos(x))dx$. ($1+\frac{\sqrt{2}}{2}$)

21. Please calculate the integration $\int\frac{x}{x^2+1}dx$ and $\int_0^{x'}(2x+2)\cos(x^2+2x)\,dx$.

($\frac{1}{2}\ln(x^2+1)+c, \sin(x'^2+2x')$)

22. Please use the relation $\frac{d\left(e^{3x}\right)}{dx}=3e^{3x}$ ($3e^{3x}dx=d(e^{3x})$) to calculate the integral of $\int e^{3x}dx$. ($\frac{e^{3x}}{3}+c$)

23. Please use the relation $\frac{d\left(\sqrt{1+x^2}\right)}{dx}=\frac{x}{\sqrt{1+x^2}}$ ($\frac{x}{\sqrt{1+x^2}}dx=d\left(\sqrt{1+x^2}\right)$) to calculate the integral of $\int\frac{x}{\sqrt{1+x^2}}dx$. ($\sqrt{1+x^2}+c$)

24. Please use the relation $\cos(x)\,dx=d(\sin(x))$ to calculate $\int_{x=0}^{x=\pi/2}(\cos(x))^3dx$.

($\frac{2}{3}$)

25. Please check that $y(x)=Ae^{-bx}$ is the solution of the 1st order differential equation $\frac{dy}{dx}+by=0$.

26. Please check that $x(t) = A\cos(\omega t + \phi)$ is a solution of the differential equation of

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

27. Please check that $N(t) = N_0 e^{-\lambda t}$ is a solution of the differential equation

$$\frac{dN(t)}{dt} = -\lambda N(t).$$

28. The rate of the population $\frac{dP}{dt}$ is proportional to its population P as $\frac{dP}{dt} = bP$.

    The initial population at $t = 0$ is P_0. Please solve the first-order differential equation $\frac{dP}{dt} = bP$ to obtain $P(t)$.