Lecture 04 Motion in Two Dimensions

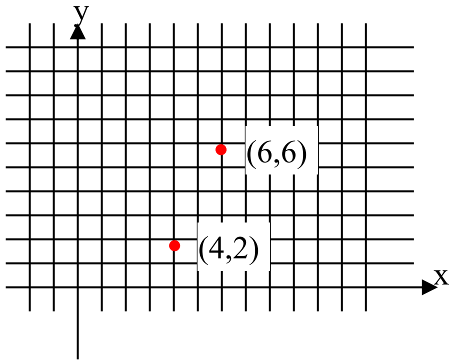
This lecture gives you the same content as that in **Chapter 03&04 in Serway/Jewett's** textbook of "Physics for Scientists and Engineers with Modern Physics".

The Displacement Vector

Coordinate Systems

$r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$, $y = r\sin\theta$

$\tan\theta = y/x$

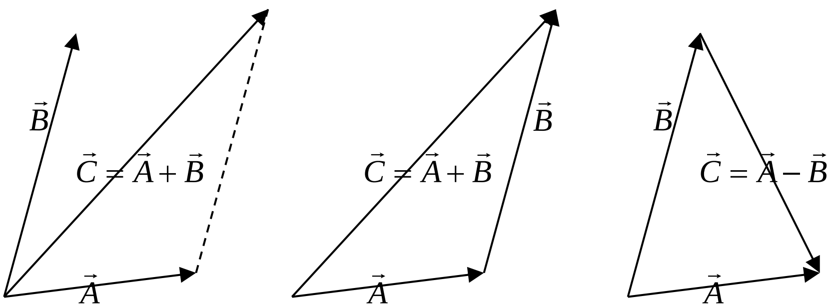


Vectors and Scalars

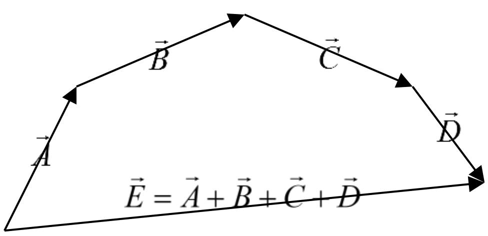
vector: displacement, $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$

scalar: distance, $|\Delta\vec{r}| = |\vec{r}_f - \vec{r}_i|$

Addition of Displacement Vectors



1. $\vec{C} = \overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{BO} = \overrightarrow{BA}$ (drawing)
2. $\vec{C} = \vec{r}_{AO} - \vec{r}_{BO} = \vec{r}_{AO} + \vec{r}_{OB} = \vec{r}_{AB}$ (relative vector)



General Properties of Vectors

Equality of Two Vectors: $\vec{A} = \vec{B}$

Adding Vectors: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

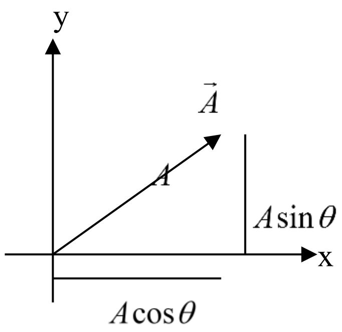
Multiplying a Vector by a Scalar: 1. $\vec{A}$ 2. $0 \cdot \vec{A} = 0$ 3. $2 \cdot \vec{A} = 2\vec{A}$

Subtracting Vectors: $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$, $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

scalar product: $\vec{A} \cdot \vec{B}$ see Chapter 6-2

vector product: $\vec{A} \times \vec{B}$ see Chapter 10-1



Components of a Vector and Unit Vectors

$\vec{A} = A_x\hat{i} + A_y\hat{j} = (A_x, A_y) = A\cos\theta\hat{i} + A\sin\theta\hat{j}$

$|\vec{A}| = \sqrt{A_x^2 + A_y^2} = A$

If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

See active figure AF0316.

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_x\hat{k}$ & $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_x\hat{k}$

--> $\vec{A} + \vec{B} = ?$, $0 \cdot \vec{A} = ?$, $2 \cdot \vec{A} = ?$, $\vec{A} - 2\vec{B} = ?$

Unit Vectors

If $A = 1$, $\vec{A} = A_x\hat{i} + A_y\hat{j}$ is a unit vector.

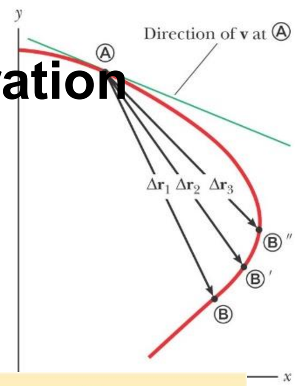
If $A \neq 1$, $\vec{A}/A$ is a unit vector. --> $\left|\frac{\vec{A}}{A}\right| = \frac{A}{A} = 1$

If $\vec{A} = A_x\hat{i} + A_y\hat{j}$ is a unit vector, $\vec{A}$ can be rewritten as $\vec{A} = \cos\theta\hat{i} + \sin\theta\hat{j}$.

Example: The resultant displacement: Δr_1=(1.5, 3), Δr_2=(2.3, 1.4), Δr_3=(-1.3, 1.5), R=?

Unit vector?

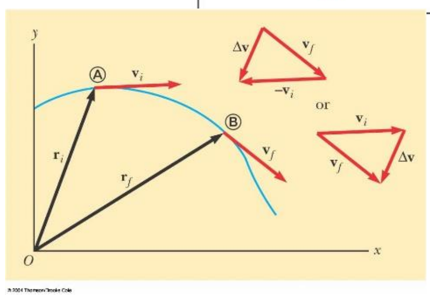
4.1 Position, Velocity, and Acceleration



displacement: $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$

average velocity: $\vec{v}_{avg} = \dfrac{\Delta\vec{r}}{\Delta t}$

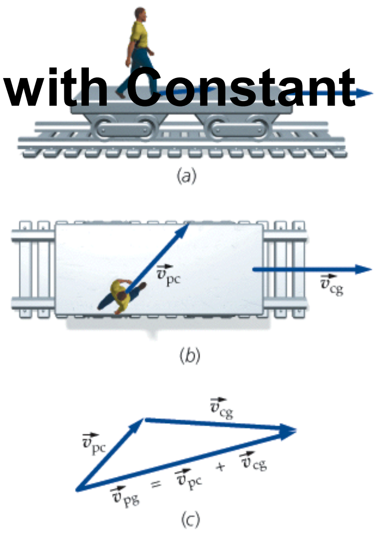
instantaneous velocity: $\vec{v} = \lim\limits_{\Delta t -> 0} \dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{d}{dt}\vec{r}$



average acceleration: $\vec{a}_{avg} = \dfrac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \dfrac{\Delta\vec{v}}{\Delta t}$

instantaneous acceleration: $\vec{a} = \lim\limits_{\Delta t -> 0} \dfrac{\Delta\vec{v}}{\Delta t} = \dfrac{d}{dt}\vec{v}$

4.2 Two Dimensional Motion with Constant Acceleration



$\vec{r} = x \cdot \hat{i} + y \cdot \hat{j}$

$\vec{v} = \dfrac{d}{dt}\vec{r} = \dfrac{d}{dt}x \cdot \hat{i} + \dfrac{d}{dt}y \cdot \hat{j}$

$\vec{v}_f = \vec{v}_i + \vec{a}t$, $v_{fx}\hat{i} + v_{fy}\hat{j} = v_{ix}\hat{i} + v_{iy}\hat{j} + a_x t \cdot \hat{i} + a_y t \cdot \hat{j}$

$\vec{r}_f = \vec{r}_i + \vec{v}_i t + \dfrac{1}{2}\vec{a}t^2$ -> separate into x and y directions

Example: A particle move through the origin of an xy coordinate system at t = 0 with initial velocity $\vec{v} = 20\hat{i} - 15\hat{j}$ (m/s). The particle moves in the xy plane with an acceleration $\vec{a} = 4\hat{i}$ (m/s^2). Determine the components of velocity as a function of time and the total velocity vector at any time.

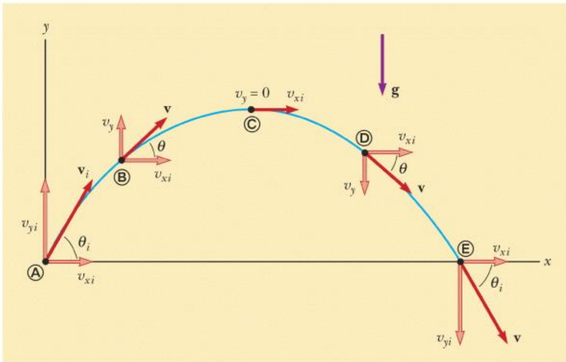
At t = 0, $\vec{x}_0 = 0\hat{i} + 0\hat{j}$, $\vec{v}_0 = 20\hat{i} - 15\hat{j}$, $\vec{a} = 4\hat{i}$

$$\vec{a} = \frac{d\vec{v}}{dt} = 4\hat{i}\,,\quad d\vec{v} = 4dt\hat{i}\,,\quad \int_{v0}^{v} d\vec{v} = \int_{0}^{t} 4dt\hat{i}\,,\quad \vec{v} = \vec{v}_0 + 4t\hat{i} = (20+4t)\hat{i} - 15\hat{j}$$

$$d\vec{x} = [(20+4t)\hat{i} - 15\hat{j}]dt\,,\quad \int_{x0}^{x} d\vec{x} = \int (20+4t)dt\hat{i} + \int_{0}^{t} (-15)dt\hat{j}\,,\quad \vec{x} = (20t + 2t^2)\hat{i} - 15t\hat{j}$$

4.3 Projectile Motion

Serway/Jewett; Principles of Physics, 3/e
Figure 3.5



We assume that (1) The free-fall acceleration $\vec{g}$ is constant over the range of motion. (2) The effect of air resistance is negligible.

$$\tan\theta_i = \frac{v_{iy}}{v_{ix}}$$

$$v_{ix} = v_i \cos\theta\,,\quad v_{iy} = v_i \sin\theta$$

$$a_x = 0\,,\quad a_y = -g$$

$$v_x = v_i \cos\theta\,,\quad v_y = v_i \sin\theta - gt$$

$$x_f = x_i + v_i \cos\theta \cdot t\,,\quad y_f = y_i + v_i \sin\theta \cdot t - \frac{1}{2}gt^2 \;\; \text{->} \;\; \vec{r}_f = \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{g}t^2$$

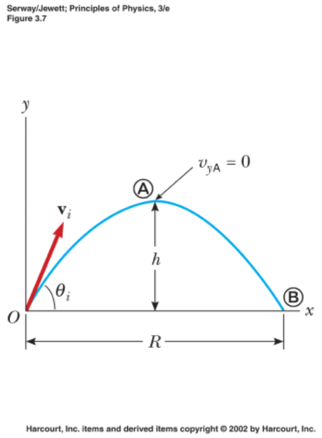
If $x_i = y_i = 0$, we have $x_f = v_i \cos\theta t$. Remove the time, we have an expression of x and y as $y_f = (\tan\theta)x - \dfrac{g}{2v_i^2\cos^2\theta_i}x^2$.

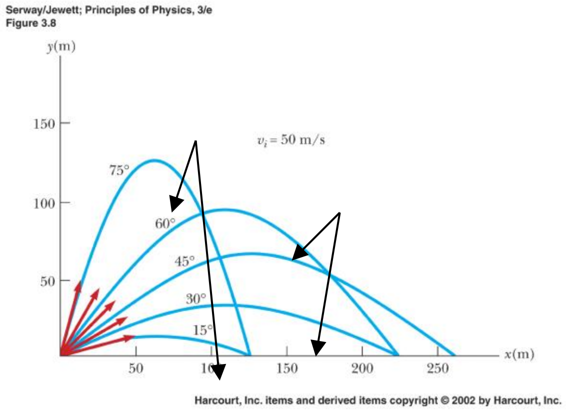
The projectile motions can be considered to be the superposition of (1) constant-velocity motion in the horizontal direction and (2) free-fall motion in the vertical direction.

Horizontal range and maximum height of a projectile

$$t_M = \frac{v_i \sin\theta}{g}\,,\quad h = \frac{v_i^2 \sin^2\theta}{2g}$$

$$R = (v_i\cos\theta)2t_M = \frac{v_i^2 \sin 2\theta}{g}$$





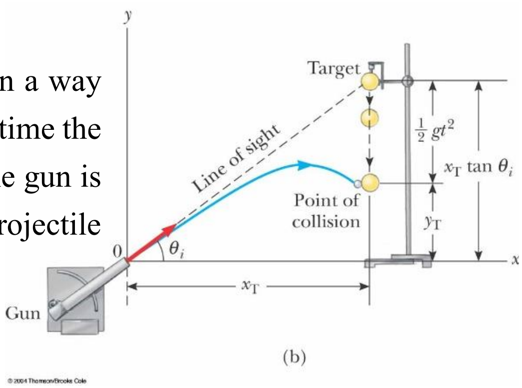
Example: A long-jumper leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s. How far does he jump in the horizontal direction? What is the maximum height reached?

$$t_M = \frac{11.0 * \sin(20.0^o)}{g} = 0.38\,\text{s}$$

$$R = 11.0 * \cos(20.0^o) * 2 * 0.38 = 7.86 \;\; \text{m}$$

$$H = \frac{1}{2} 9.8 * (0.38)^2 = 0.71\text{m}$$

Example: A projectile is fired at a target T in a way that the projectile leaves the gun at the same time the target is dropped from rest. Show that if the gun is initially aimed at the stationary target, the projectile hits the target.



Prove that $\tan\theta_i = \frac{v_{yi}}{v_{xi}} = \frac{y_T}{x_T}$.

$$x_T = v_{xi} t \;\; \text{->} \;\; t = \frac{x_T}{v_{xi}}, \text{ Hit means that } \; y_T = v_{yi}\frac{x_T}{v_{xi}} - \frac{1}{2} g \left(\frac{x_T}{v_{xi}}\right)^2 + \frac{1}{2} g \left(\frac{x_T}{v_{xi}}\right)^2.$$

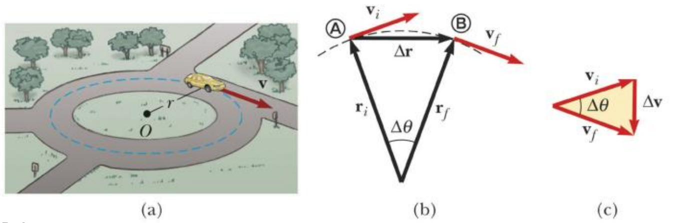
Example: A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s. If the height of the building is 45.0 m, how long does it take to reach the ground?

4.4 Uniform Circular Motion

Uniform Circular Motion

centripetal acceleration, use polar coordinate to solve this typical problem

$$T = \frac{2\pi r}{v}, \quad \omega = \frac{d\theta}{dt} = \frac{2\pi}{T} = \frac{v}{r}$$



(a) (b) (c)

(1) $\vec{r} = r\cos\theta \cdot \hat{i} + r\sin\theta \cdot \hat{j}$

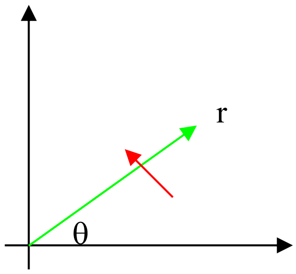
$$\vec{v} = -r\sin\theta \frac{d\theta}{dt}\hat{i} + r\cos\theta \frac{d\theta}{dt}\hat{j}$$

$$= -v\sin\theta \cdot \hat{i} + v\cos\theta \cdot \hat{j}$$

$$\vec{a} = -v\cos\theta \frac{d\theta}{dt}\hat{i} - v\sin\theta \frac{d\theta}{dt}\hat{j}$$

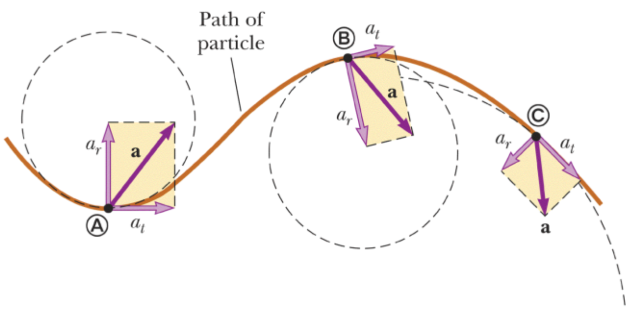
$$= -\frac{v^2}{r}\cos\theta \cdot \hat{i} - \frac{v^2}{r}\sin\theta \cdot \hat{j}$$

$$a = \frac{v^2}{r}$$



(2) $\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}, \quad \frac{|\Delta\vec{v}|}{|\vec{v}|} = \frac{|\Delta\vec{r}|}{|\vec{r}|}, \quad |\vec{a}_{avg}| = \frac{v}{r}\frac{\Delta r}{\Delta t}, \quad \lim_{\Delta t \to 0}\frac{\Delta r}{\Delta t} = v, \quad a = \frac{v^2}{r}$

4.5 Tangential and Radial Acceleration



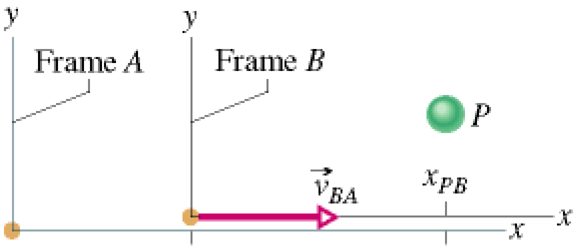
the change in speed: $a_t = \frac{d}{dt}|\vec{v}|$, the change in direction: $a_r = -\frac{v^2}{r}$

$$a = \sqrt{a_t^2 + a_r^2}$$

Example: A car exhibits a constant acceleration of 0.300 m/s^2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like a circle of radius 500 m. At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of 6.00 m/s. What is the direction and the magnitude of the total acceleration vector for the car at this moment?

$$\vec{a}_t = 0.300\hat{x} \quad (\text{m/s}^2) \quad \vec{a}_r = -\frac{6^2}{500}\hat{y} = -0.072\hat{y} \quad (\text{m/s}^2)$$

4.6 Relative Velocity and Relative Acceleration



$$\vec{r}_{PO} = \vec{r}_{PO'} + \vec{v}_{O'O}t\,, \quad \vec{v}_{PO} = \vec{v}_{PO'} + \vec{v}_{O'O}$$

$$x_{PA} = x_{PB} + x_{BA}\,, \quad \frac{d}{dt}x_{PA} = \frac{d}{dt}x_{PB} + \frac{d}{dt}x_{BA}\,, \quad v_{PA} = v_{PB} + v_{BA}$$

$$\frac{d}{dt}v_{PA} = \frac{d}{dt}v_{PB} + \frac{d}{dt}v_{BA}\,, \quad a_{PA} = a_{PB}$$

Relative Velocity

$$\vec{r}_{PO} = \vec{r}_{PO'} + \vec{v}_{O'O}t\,, \quad \vec{v}_{PO} = \vec{v}_{PO'} + \vec{v}_{O'O}$$

$\vec{x}_{PA}$ & $\vec{x}_{BA}$ relative to reference A

$\vec{x}_{PB}$ relative to reference B

If the particle P and Frame B move at a speed of several tenths of light speed, the special relativity of modern Physics must be adopted for relative velocity.

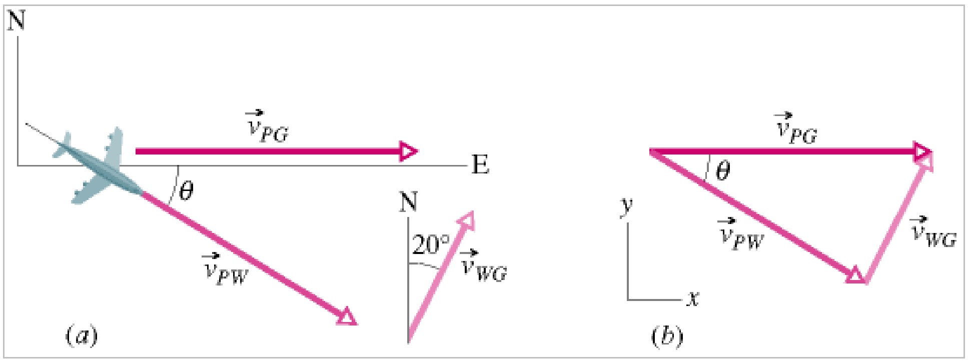
$v_{PB} = 0.6C\,,\ v_{BA} = 0.6C$ → $v_{PA} = v_{PB} + v_{BA} = 1.2C$???

$$v_{PA} = \frac{v_{PB} + v_{BA}}{1 + v_{PB}v_{BA}/c^2} = \frac{1.2C}{1 + 0.36} = \frac{1.2}{1.36}C < C$$

Example: Barbara's velocity relative to Alex is a constant v_{BA} = 52 km/h and car P is moving in the negative direction of the x axis. If Alex measures a constant velocity v_{PA} = -78 km/h for car P, what velocity v_{PB} will Barbara measure?

$$v_{BA} = 52\text{ km/h}, \quad v_{PA} = -78\text{ km/h}, \quad v_{PB} = v_{PA} + v_{AB} = v_{PA} - v_{BA} = -78 - 52 = -130\text{ km/h}$$

Example: A plane moves due east (directly toward the east) while the pilot points the plane somewhat south of east, toward a steady wind that blows to the northeast. The plane has velocity V_{PW} relative to the wind, with an airspeed (speed relative to the wind) of 215 km/h, directed at angle θ south of east. The wind has velocity V_{WG} relative to the ground, with a speed of 65.0 km/h, directed 20.0° east of north. What is the magnitude of the velocity V_{PG} of the plane relative to the ground, and what is theta?



$v_{PW} = 215\cos\theta\hat{i} - 215\sin\theta\hat{j}$, $v_{WG} = 65\sin 20^o\,\hat{i} + 65\cos 20^o\,\hat{j}$, $v_{PG} = v\hat{i} + 0\,\hat{j}$

$v_{PW} + v_{WG} = v_{PG}$ -> $\left(215\cos\theta + 65\sin 20^o\right)\hat{i} + \left(-215\sin\theta + 65\cos 20^o\right) = v\hat{i} + 0\,\hat{j}$

$\theta = 16.5^o$, v = 228.4 km/h