Lecture 10 Rotation of a Rigid Object About a Fixed Axis

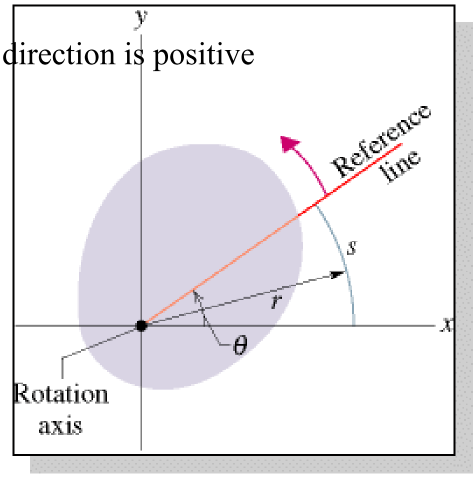
10.1 Angular Position, Velocity, and Acceleration

Translation: motion along a straight line (circular motion ?)
Rotation: surround itself, spins
rigid body: no elastic, no relative motion
rotation: moving surrounding the fixed axis, rotation axis, axis of rotation

Angular position: $\theta = \frac{s}{r}$, $1rev = 360^o = 2\pi(rad)$, $1rad = \frac{360}{2\pi} = 57.3^o$

Angular displacement: $\Delta\theta = \theta_1 - \theta_2$

An angular displacement in the counterclockwise direction is positive



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Angular velocity: averaged $\omega = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$

instantaneous: $\omega = \frac{d\theta}{dt}$, rpm = rev per minute

Angular acceleration: averaged $\alpha = \frac{\Delta\omega}{\Delta t}$

instantaneous: $\alpha = \frac{d\omega}{dt}$

Sample Problem:

The disk in Fig. 11-5a is rotating about its central axis like a merry-go-round. The angular position θ(t) of a reference line on the disk is given by
$\theta = -1.00 - 0.600t + 0.250t^2$ with t in seconds, q in radians, and the zero angular position as indicated in the figure.

a) Graph the angular position of the disk versus time from t = -3.0 s to t = 6.0 s. Sketch the disk and its angular position reference line at t = -2.0 s, 0 s, and 4.0 s, and when the curve crosses the t axis.
b) At what time t_{min} does $\theta(t)$ reach the minimum value? What is that minimum value?
c) Graph the angular velocity ω of the disk versus time from t = -3.0 s to t = 6.0 s. Sketch the disk and indicate the direction of turning and the sign of ω at t = -2.0 s and 4.0 s, and also at t_{min}.

Right-hand rule

Specify Your Axis

10.2 Rotational Kinematics: The Rigid Object Under Constant Angular Acceleration

Serway/Jewett; Principles of Physics, 3/e
Table 10.1

TABLE 10.1 A Comparison of Equations for Rotational and Translational Motion: Kinematic Equations

Rotational Motion About a Fixed Axis with α = Constant (Variables: θ_f and ω_f)	Translational Motion with a = Constant (Variables: x_f and v_f)
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$

Sample Example: A grindstone rotates at constant angular acceleration $\alpha = 0.35$ rad/s^2. At time $t = 0$, it has an angular velocity of $\omega_0 = -4.6$ rad/s and a reference line on it is horizontal, at the angular position $\theta_0 = 0$.

$$\alpha = \frac{d\omega}{dt} = 0.35 rad/s^2, \quad d\omega = 0.35 dt, \quad \omega - \omega_0 = 0.35t, \quad \omega = -4.6 + 0.35t$$

$$\frac{d\theta}{dt} = \omega = -4.6 + 0.35t, \quad \theta - \theta_0 = -4.6t + 0.175t^2, \quad \theta = -4.6t + 0.175t^2$$

a) At what time after $t = 0$ is the reference line at the angular position $\theta = 5.0$ rev?

c) At what time t does the grindstone momentarily stop?

10.3 Angular and Translational Quantities

The position: $s = r\theta$

The speed (velocity?): $v = r\omega$, $T = \frac{2\pi r}{v}$

The acceleration: $a_t = r\alpha$, $a_r = \frac{v^2}{r} = r\omega^2$

Sample Example:

Figure shows a centrifuge used to accustom astronaut trainees to high accelerations. The radius *r* of the circle traveled by an astronaut is 15 m.

(a) At what constant angular speed must the centrifuge rotate if the astronaut is to have a linear acceleration of magnitude 11*g*?

$$r\omega^2 = 15 \cdot \omega^2 = 11 \times 9.8, \quad \omega = \sqrt{\frac{11 \times 9.8}{15}} = 2.68 rad/s$$

(b) What is the tangential acceleration of the astronaut if the centrifuge accelerates at a constant rate from rest to the angular speed of (a) in 120 s?

$$a_t = r\alpha = r\alpha_{avg} = 15\frac{\omega - 0}{120} = 15\frac{2.68}{120} = 0.34 m/s^2$$

10.4 Rotational Kinetic Energy

$$K = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum m_i \omega^2 r_i^2 = \frac{1}{2}\sum m_i r_i^2 \omega^2$$

$I = \sum m_i r_i^2$ (rotational inertia), $K = \frac{1}{2} I\omega^2$

Example: The Oxygen Molecule

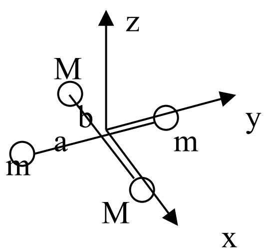
Consider an oxygen molecule (O_2) rotating in the xy plane about the z axis. The rotation axis passes through the center of the molecule, perpendicular to its length. The mass of each oxygen atom is 2.66 x 10^{-26} kg, and at room temperature the average separation between the two atoms is d = 1.21 x 10^{-10} m. (The atoms are modeled as particles.) (a) Calculate the moment of inertia of the molecule *about the z axis.*

$I = \sum m_i r_i^2$ what is the radius r_i? $I = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2$

(b) If the angular speed of the molecule about the z axis is 4.60 X 10^{12} rad/s, what is its rotational kinetic energy?

$E_k = \frac{1}{2}mv^2$ -> $E_R = \frac{1}{2}I\omega^2$

Example: Four Rotating Object

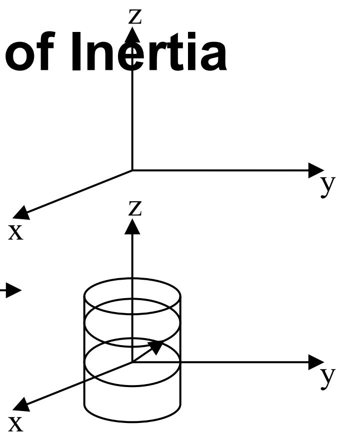


Calculate the moment of inertia if the rotation is about the z axis and about the y axis.

$I = \sum m_i r_i^2$ what is the radius r_i? $I_z = ma^2 + ma^2 + Mb^2 + Mb^2$

$I_y = Mb^2 + Mb^2$

10.5 Calculation of Moments of Inertia

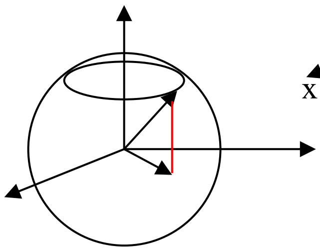


Cartesian Coordinate: x, y, z

Spherical Coordinate: r, θ, ϕ

$z = r\cos\theta$

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi$



Cylindrical Coordinate: r, θ, z

z

$x = r\cos\theta, \quad y = r\sin\theta$

(a)

$M = 2\pi R\rho$

$I = \int R^2 \rho R d\theta = MR^2$

(b)

$M = \pi(R_2^{\,2} - R_1^{\,2})\rho$

$I = \int_{R_1}^{R_2} r^2 \int_0^{2\pi} r\rho d\theta dr = \int_{R1}^{R2} 2\pi\rho r^3 dr = \frac{M}{2}(R_1^{\,2} + R_2^{\,2})$

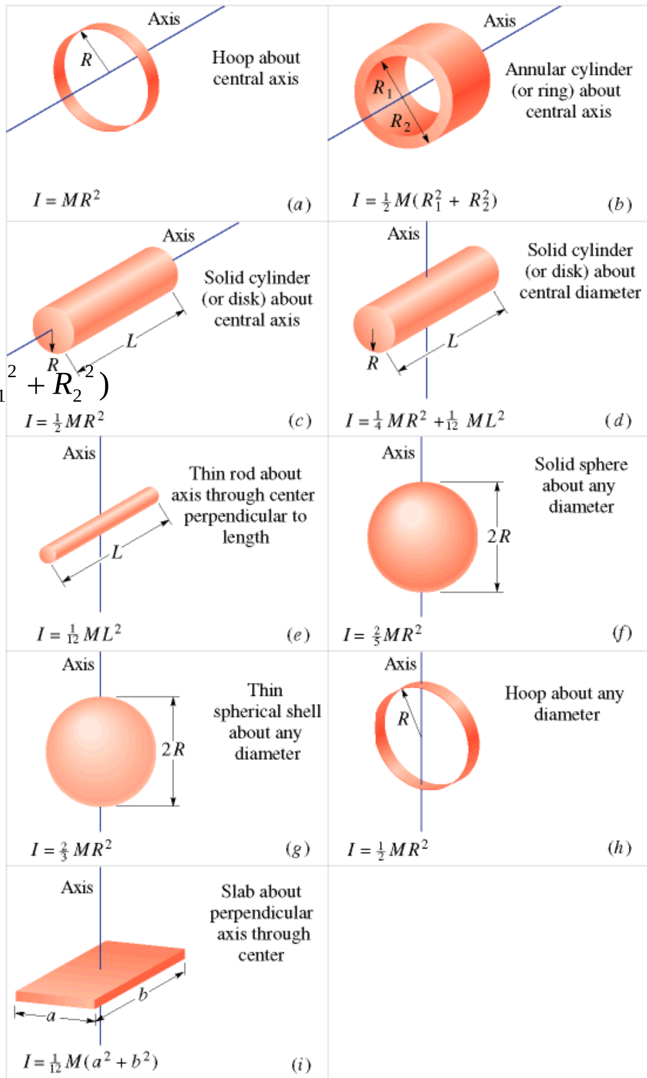
(c)

$M = \pi R^2 L\rho$

$I = \int_0^L \int_0^R r^2 \int_0^{2\pi} r\rho d\theta dr dz$

$I = L\int_0^R 2\pi r^3 \rho dr = \frac{1}{2}MR^2$

(d)



$$M = \pi R^2 L\rho$$

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}}\int_0^R\int_0^{2\pi}\left(z^2 + r^2\sin^2\theta\right)\rho r d\theta dr dz = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

(e) from (d), $R << L$, $I = \frac{1}{12}ML^2$

(f)

$$M = \frac{4\pi}{3}R^3\rho$$

$$I = \int_0^R\int_0^\pi\int_0^{2\pi} r^2\sin^2\theta\rho r\sin\theta d\phi r d\theta dr = \frac{2}{5}MR^2$$

(g)

$$I = \sigma\int_0^\pi (R\sin\theta)^2 2\pi R^2\sin\theta d\theta = \frac{M}{4\pi R^2}\int_0^\pi (R\sin\theta)^2 2\pi R^2\sin\theta d\theta = \frac{M}{2}\int_{-1}^{1}\left(1-\cos^2\theta\right)R^2 d(\cos\theta) = \frac{2}{3}MR^2$$

(h)

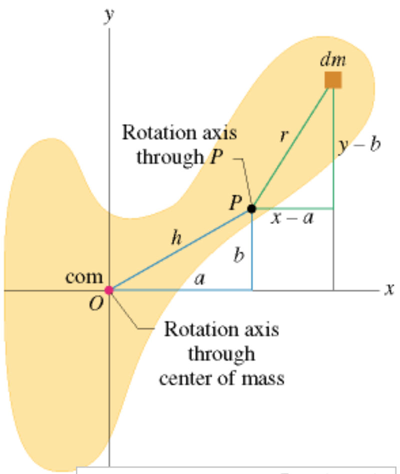
$$M = 2\pi R\rho$$

$$I = \int R^2\sin^2\theta\rho R d\theta = MR^2 = \frac{1}{2}MR^2$$

(i)

$$I = \frac{M}{ab}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left(x^2 + y^2\right)dxdy = \frac{1}{12}M(a^2 + b^2)$$

Parallel axis theorem:



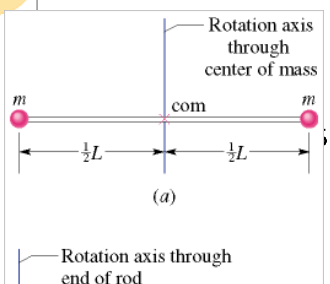
$$I = I_{CM} + Mh^2$$

$$I = \int r^2 dm = \int\left[(x-a)^2 + (y-b)^2\right]dm$$

$$I = \int\left(x^2 + y^2\right)dm - 2a\int xdm - 2b\int ydm + \int\left(a^2 + b^2\right)dm$$

$$I = I_{cm} + Mh^2$$

Sample Example:



(a) What is the rotational inertia I_{com} of this body about an axis through its center of mass, perpendicular to the rod as shown?

$$I = \sum m_i r_i^2 = m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{2}mL^2$$

(b) What is the rotational inertia I of the body about an axis through the left end of the rod and parallel to the first axis?

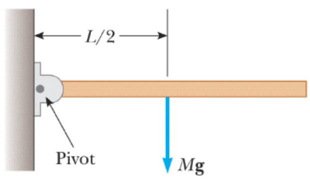
$$I = \sum m_i r_i^2 = m \cdot 0^2 + mL^2 = mL^2$$

use parallel axis theorem: $I = I_{CM} + M\left(\frac{L}{2}\right)^2 = \frac{1}{2}mL^2 + 2m\left(\frac{L}{2}\right)^2 = mL^2$

Example: Rotating Rod

A uniform rod of length L and mass M is free to rotate on a frictionless pin through one end. The rod is released from rest in the horizontal position. (a) What is the angular speed of the rod at its lowest position?

Serway/Jewett; Principles of Physics, 3/e
Figure 10.10

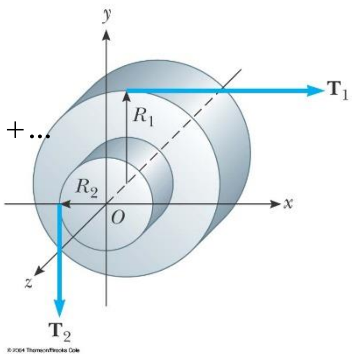


1. $I_{cm} = \frac{1}{12}ML^2$, $I = I_{cm} + M(\frac{L}{2})^2 = \frac{1}{3}ML^2$, Energy conservation: $\frac{1}{2}I\omega^2 = Mgh$,

   $\frac{1}{2}\frac{1}{3}ML^2\omega^2 = Mg\frac{L}{2}$, $\omega = \sqrt{\frac{3g}{L}}$, $v_{cm} = \frac{1}{2}L\omega$

2. $Mgh = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}M{v_{cm}}^2$

10.6 Torque



Torque is a vector: $N = \tau = \vec{r} \times \vec{F} = rF\sin\phi$, $\tau = \tau_1 + \tau_2 + \tau_3 + \ldots$

Torque <-> Force, What is the difference?

Torque can increase or decrease the angular velocity.

What about force?

Example: The net torque on a cylinder

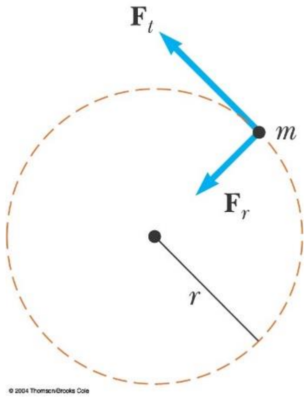
$$\vec{\tau} = (R_2T_2 - R_1T_1)\hat{k}$$

10.7 The Rigid Body Under a Net Torque

$F_t = F_{\tan gent}$, $F_r = F_{along\_the\_radius}$

$F_t = F_{\tan gent}$ is used for angular acceleration

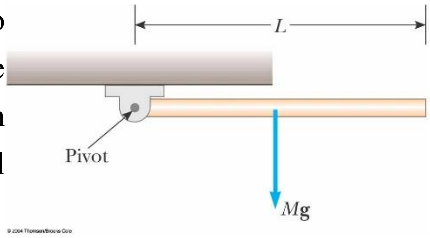
$F_t = ma_t = mr\alpha$, $\tau = rF_t = mr^2\alpha = I\alpha$



The torque acting on the particle is proportional to its angular acceleration.

Example: Rotating Rod

A uniform rod of length L and mass M is attached to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position. What is the initial angular acceleration?

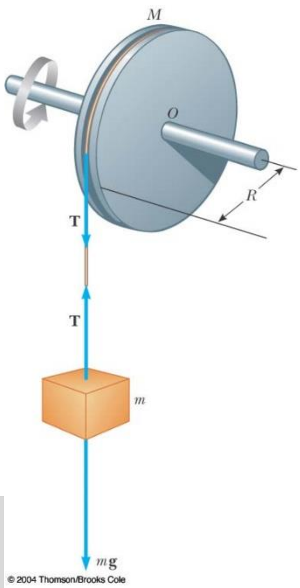


$$I_{CM} = \frac{1}{12}ML^2,\ I_{pivot} = I_{CM} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$$

$$\tau = \frac{L}{2}Mg = I\alpha,\ \alpha = \frac{3g}{2L},\ a_t = L\alpha = \frac{3}{2}g \text{ at right end}$$

Example: Angular Acceleration of a Wheel

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless horizontal axle. A light cord wrapped around the wheel supports an object of mass m. Calculate the angular acceleration of the wheel, the linear acceleration of the object, and the tension in the cord.

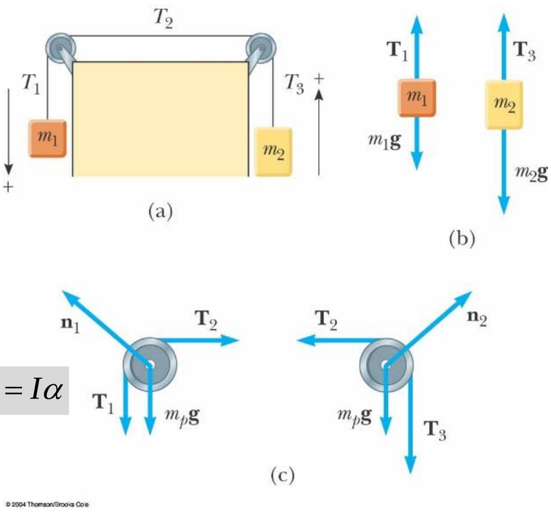


$$I = \frac{1}{2}MR^2,\ mg - T = ma,\ \tau = RT = I\alpha,\ a = R\alpha$$

$$mg - \frac{I}{R}\frac{a}{R} = ma,\ a = \frac{mg}{m + \frac{I}{R^2}},\ T = \frac{I\alpha}{R} = \frac{Ia}{R^2} = \frac{mgI}{mR^2 + I},\ \alpha =$$

Example: Atwood Machine Revisited

Two blocks having masses m1 and m2 are connected to each other by a light cord that passes over two identical frictionless pulleys, each having a moment of inertia I and radius R. Find the acceleration of each block and the tension T1, T2, and T3 in the cord. (No slipping)

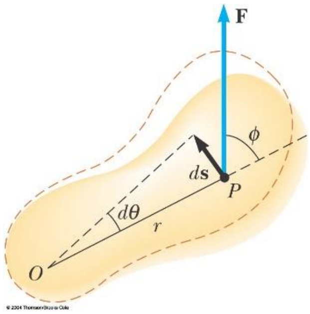


$m_1 g - T_1 = m_1 a$, $R(T_1 - T_2) = I\alpha$, $R(T_2 - T_3) = I\alpha$

$T_3 - m_2 g = m_2 a$, $R\alpha = a$

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + \dfrac{2I}{R^2}}$$

10.8 Energy Considerations in Rotational Motion



$$dW = \vec{F} \cdot d\vec{s} = (F \sin\phi) r d\theta$$

$$dW = \tau \cdot d\theta$$

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$

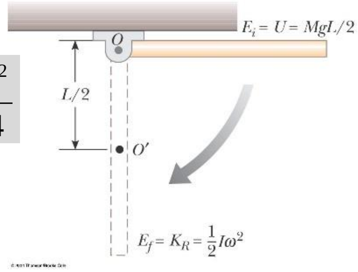
$$\sum \tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega$$

work-kinetic energy theorem: $\sum \tau \cdot d\theta = I\omega d\omega$, $\sum W = \int_{\omega i}^{\omega f} I\omega d\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$

Rotational motion	Linear motion
$W = \int_{\theta i}^{\theta f} \tau \cdot d\theta$	$W = \int_{xi}^{xf} F_x dx$
$K_R = \frac{1}{2}I\omega^2$	$K = \frac{1}{2}mv^2$
$P = \tau\omega$	$P = Fv$
$p = I\omega$	$p = mv$
$\sum \tau = \frac{dL}{dt}$	$\sum F = \frac{dp}{dt}$

Example: Rotating Rod Revisited

A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end. The rod is released from rest in the horizontal position. Determine the tangential speed of the CM and the tangential speed of the lowest point on the rod when it is in the vertical position.



$$Mgh = Mg\frac{L}{2} = \frac{1}{2}I\omega^2, \quad I_{CM} = \frac{1}{12}ML^2, \quad I_{pivot} = I_{CM} + M\frac{L^2}{4}$$

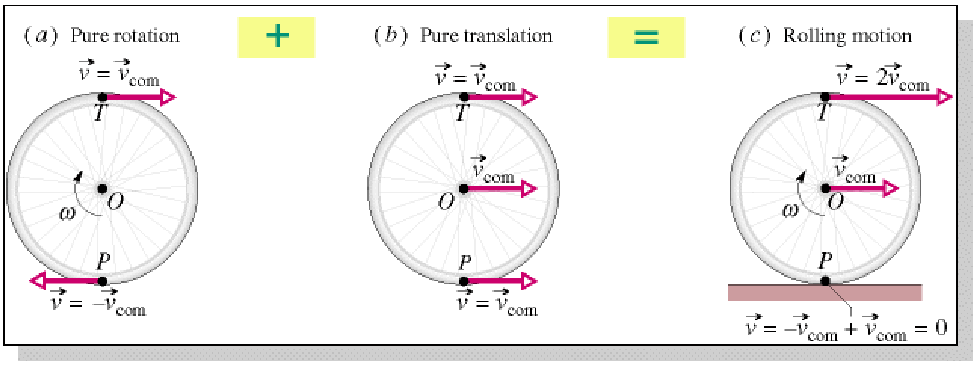
$$Mg\frac{L}{2} = \frac{1}{2}\frac{ML^2}{3}\omega^2, \quad \omega = \sqrt{\frac{3g}{L}}$$

$$v_{CM} = \frac{L}{2}\omega = \frac{\sqrt{3gL}}{2}$$

$$v_{end} = L\omega = \sqrt{3gL}$$

10.9 Rolling Motion of a Rigid Object

Rolling as rotation and translation combined



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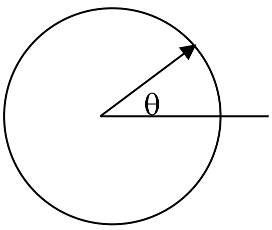
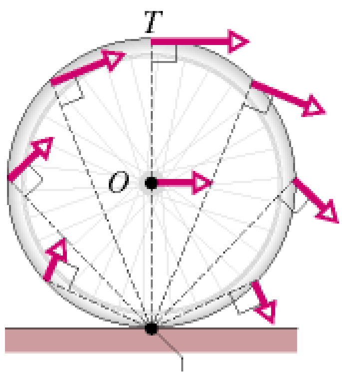


$$v_{cm} = R\omega, \quad v_{edge} = R\omega\sin\theta\cdot\hat{i} - R\omega\cos\theta\cdot\hat{j}$$

$$v_{edge} = R\omega(1+\sin\theta)\hat{i} - R\omega\cos\theta\cdot\hat{j}$$

Rolling as pure rotation



Rotation axis at P

1. $K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

2. $K = \frac{1}{2}I_p\omega^2$, $I_P = I_{CM} + MR^2$, $K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}MR^2\omega^2$

Sample Problem:

A uniform solid cylindrical disk, of mass M = 1.4 kg and radius R = 8.5 cm, rolls smoothly across a horizontal table at a speed of 15 cm/s. What is its kinetic energy K?

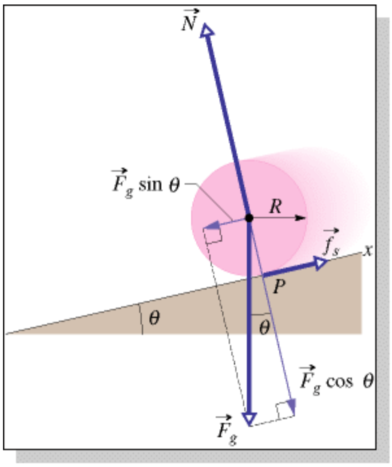
$$K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}Mv_{CM}^2$$

$$= \frac{1}{2}\frac{MR^2}{2}\left(\frac{v_{CM}}{R}\right)^2 + \frac{1}{2}Mv_{CM}^2 = \frac{3}{4}Mv_{CM}^2 = 24$$

$$mg\sin\theta - f_s = ma, \quad a = R\alpha$$

$$I\alpha = R \cdot f_s, \quad f_s = I\frac{a}{R^2}$$

$$a = \frac{mg\sin\theta}{m + \frac{I}{R^2}}$$



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Sample Example:

A uniform ball, of mass *M* = 6.00 kg and radius *R*, rolls smoothly from rest down a ramp at angle θ = 30.0°.

(a) The ball descends a vertical height *h* = 1.20 m to reach the bottom of the ramp. What is its speed at the bottom?

$$\frac{1}{2}Mv_{CM}{}^2 + \frac{1}{2}I_{CM}\omega^2 = Mgh, \quad I = \frac{2}{5}MR^2, \quad R\omega = v_{CM}$$

$$\frac{1}{2}Mv_{CM}{}^2 + \frac{1}{2}\frac{2}{5}MR^2\frac{v_{CM}{}^2}{R^2} = Mgh, \quad v_{CM} = \sqrt{\frac{10gh}{7}} = 4.1m/s$$

(b) What are the magnitude and direction of the friction force on the ball as it rolls down the ramp?

$$a = \frac{g\sin\theta}{1 + \frac{I}{MR^2}} = \frac{4.9}{1 + \frac{2}{5}} = 3.5m/s^2, \quad f_s = \frac{I}{R^2}a = \frac{2}{5}Ma = 8.4N$$