Chapter 31 Inductance

31.1 Self-Induction and Inductance

Self-Inductance

$\Phi_m = BA = \mu_0 nIA \propto I$ --> $\Phi_m = LI$

The unit of the inductance is henry (H). $1\text{H} = 1\frac{\text{Wb}}{\text{A}} = 1\frac{\text{T}\cdot\text{m}^2}{\text{A}}$

When the current in the circuit is changing, the magnetic flux is also changing.

$\frac{d\Phi}{dt} = \frac{d(LI)}{dt} = L\frac{dI}{dt}$ --> The induced emf should be $\varepsilon = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$

The self inductance of a infinite long solenoid:

$\Phi_m = NAB = NA\mu_0\frac{N}{l}I = \mu_0 A\frac{N^2}{l}I$ --> $L = \frac{\Phi_m}{I} = \mu_0 A\frac{N^2}{l}$

$\varepsilon = -L\frac{dI}{dt}$

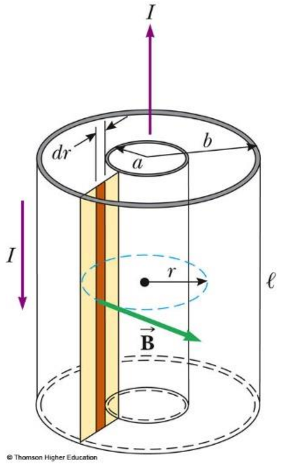
Considering the inductor having an internal resistance r, the potential difference is:

$\Delta V = \varepsilon - Ir = -L\frac{dI}{dt} - Ir$

Example: Model a long coaxial cable as two thin, concentric, cylindrical conducting shells of radii a and b and length l. The conducting shells carry the same current in opposite directions. Calculate the inductance L of this cable.

Calculate magnetic flux:

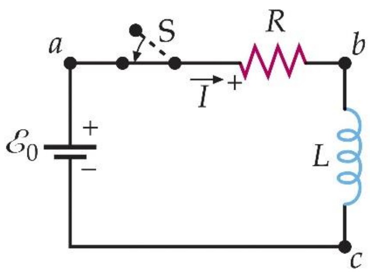
$B = \frac{\mu_0 I}{2\pi r}$ → $\Phi = \int_0^l\int_a^b \frac{\mu_0 I}{2\pi r}\,dr\,dz = \frac{\mu_0 I}{2\pi} l \ln\left(\frac{b}{a}\right)$



$$\Phi = LI \;\rightarrow\; L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)$$

31.2 RL Circuits

Use Kirchhoff's rule:



$$\oint \vec{E}\cdot d\vec{l} = 0$$

$$\varepsilon_0 - IR - L\frac{dI}{dt} = 0$$

Differential Eq: $\varepsilon_0 - IR - L\frac{dI}{dt} = 0$ --> $\varepsilon_0 - IR = L\frac{dI}{dt}$

--> $dt = L\frac{dI}{(\varepsilon_0 - IR)}$ --> $dt = -\frac{L}{R}\frac{d(\varepsilon_0 - IR)}{(\varepsilon_0 - IR)}$

--> $-\frac{R}{L}t = \ln\left(\frac{\varepsilon_0 - IR}{\varepsilon_0}\right)$ --> $\varepsilon_0 - IR = \varepsilon_0 e^{-\frac{R}{L}t}$ --> $I = \frac{\varepsilon_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

Time constant: $\tau = \frac{L}{R}$

Example: Find the total energy dissipated in the resistor R, when the current in the inductor decreases from its initial value of I_0 to 0?

$$I = \frac{\varepsilon_0}{R}e^{-\frac{R}{L}t},\quad P = I^2R \text{ --> } U = \int_0^\infty I_0^2 e^{-2\frac{R}{L}t}R\,dt = \frac{1}{2}LI_0^2$$

31.3 Energy in a Magnetic Field

Obtain the magnetic energy from the emf induced by self inductance.

$\Phi_m = LI$ --> The induced emf is $\varepsilon = -\frac{d\Phi_m}{dt} = -L\frac{dI}{dt}$

The energy dissipated or the power is $P = IV = I\varepsilon = -LI\frac{dI}{dt}$

The total energy when the current has reached its final value I_f is:

$$U = \int_{t=0}^{t=tf}\left(LI\frac{dI}{dt}\right)dt = \int_{I=0}^{I=If} LI\,dI = \frac{1}{2}LI^2$$

Calculate the magnetic energy by obtaining the energy stored in the self inductor of an infinite solenoid.

$B = \mu_0 nI$, $\Phi = nlA(\mu_0 nI) = LI$

$$U = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 (Al) I^2 = u_m V$$

$$u_m = \frac{1}{2}\mu_0 n^2 I^2 = \frac{B^2}{2\mu_0} \quad \text{<---->} \quad u_e = \frac{1}{2}\varepsilon_0 E^2$$ (Do you remember how to get this?)

Example: A certain region of space contains a uniform magnetic field of 0.020 T and a uniform electric field of 2.5 X 10^6 N/C. Find (a) the total electromagnetic density.

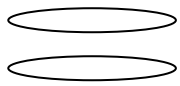
$$u_e = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\left(8.85\times10^{-12}\right)\left(2.5\times10^6\right)^2 = 27.7\text{J/m}^3$$

$$u_m = \frac{B^2}{2\mu_0} = \frac{(0.02)^2}{2\left(4\pi\times10^{-7}\right)} = 159\text{J/m}^3$$

31.4 Mutual Inductance

Mutual Inductance

The magnetic field of loop1 is: $B \sim \dfrac{\mu_0 I_1}{2R}$



The flux at 2 is $\Phi_2 \sim \dfrac{\mu_0 I_1}{2R}\pi R^2 = \mu_0 \dfrac{\pi R^2}{2} I_1 = I_1 M_{12}$

The flux at 1 is $\Phi_1 \sim \mu_0 \dfrac{\pi R^2}{2} I_2 = I_2 M_{21}$

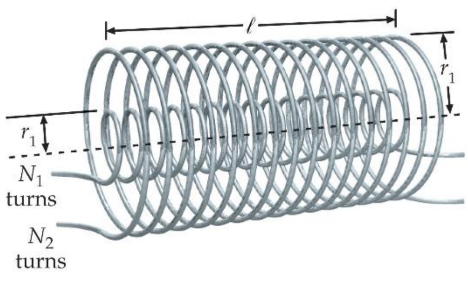
The concept of inductance: $\Phi = MI$

$M_{12} = M_{21}$ --> The mutual inductance is determined when the geometrical configuration between the two loops is given.

B in 1 due to 2: $B = \mu_0 \dfrac{N_2}{l} I_2$

Flux in 1: $\Phi = N_1\left(\pi r_1^2\right)\left(\mu_0 \dfrac{N_2}{l} I_2\right) = I_2 M_{21}$

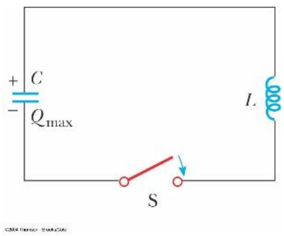
B in 2 due to 1: $B = \mu_0 \dfrac{N_1}{l} I_1$



Flux in 2: $\Phi = N_2\left(\pi r_1^2\right)\left(\mu_0 \frac{N_1}{l} I_1\right) = I_1 M_{12}$

$$M_{12} = M_{21} = \mu_0\left(\pi r_1^2\right)\frac{N_1 N_2}{l}$$

31.5 Oscillations in an LC Circuit



Kirchhoff's Rule:

$$-L\frac{dI}{dt} - \frac{Q}{C} = 0 \;\rightarrow\; L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

Compare with: $F = ma = -kx$

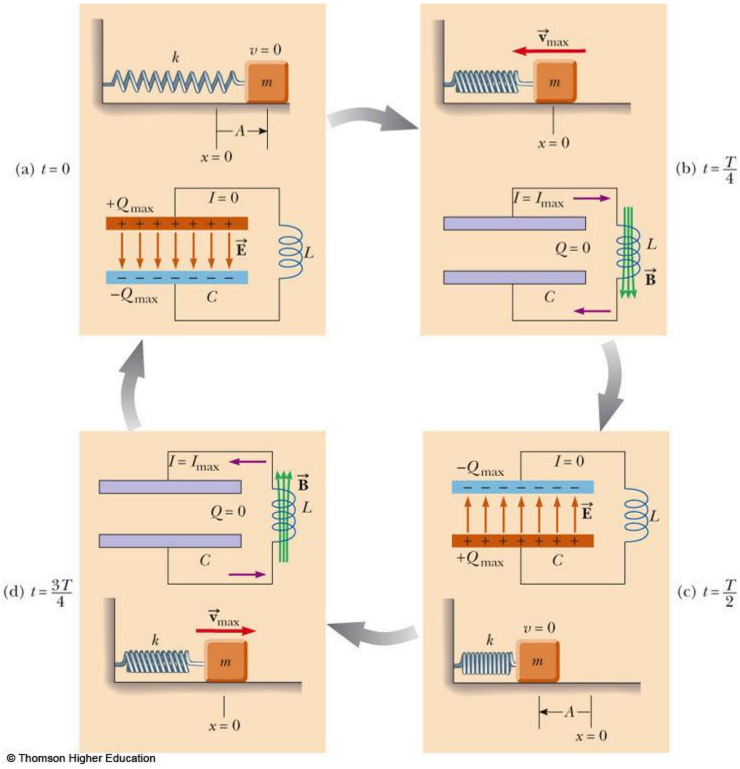
$$m\frac{d^2x}{dt^2} + kx = 0 \quad \text{solution: } x(t) = A\cos(\omega t + \phi)$$

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \quad \text{solution: } Q(t) = Q_{\max}\cos(\omega t + \phi),\quad \omega = \frac{1}{\sqrt{LC}}$$

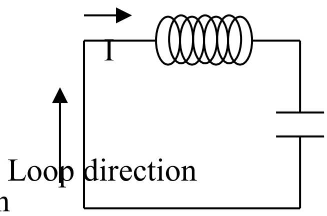
$$\rightarrow\; I = -\omega Q_{\max}\sin(\omega t + \phi)$$

$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{L}{2}I^2 = \frac{1}{2C}Q_{\max}^2\cos^2(\omega t + \phi) + \frac{L}{2}\omega^2 Q_{\max}^2\sin^2(\omega t + \phi)$$

$$= \frac{Q_{\max}^2}{2C}$$



Apply the Kirchhoff's loop rule:



$$-L\frac{dI}{dt}-\frac{Q}{C}=0$$

--> $L\frac{d^2Q}{dt^2}+\frac{Q}{C}=0$ (2nd Differential Equation, compare with harmonic oscillation: $F=ma=m\frac{d^2x}{dt^2}=-kx$ with the answer of $x=A\cos(\omega t+\delta)$, where $\omega=\sqrt{\frac{k}{m}}$)

Remember the pattern of this differential equation:

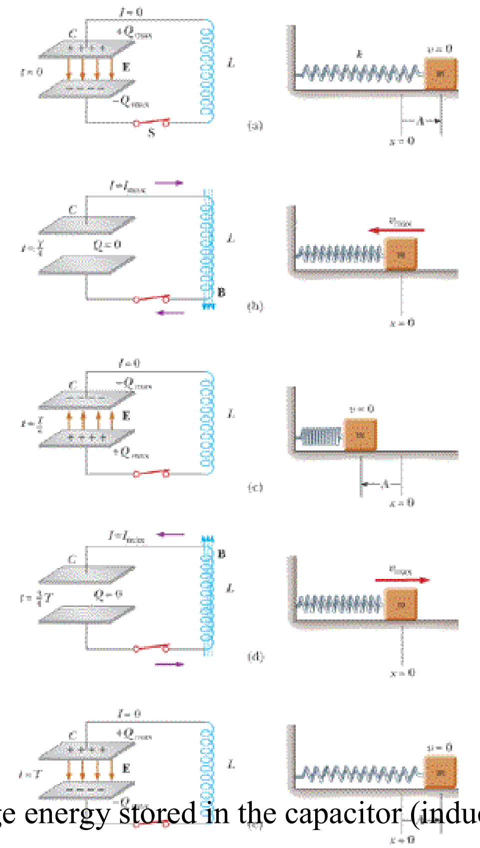
$L\frac{d^2Q}{dt^2}+\frac{Q}{C}=0$ --> the solutions are periodical functions and the useable functions are $\sin(x)$ $(\cos(x))$, $\exp(ix)$ $(\exp(x))$, …

Guess that the answer is $Q=A\cos(Bt+C)$. (Here *A* and *C* can be determined by initial conditions.)

--> $-LB^2A\cos(Bt+C)+\frac{1}{C}A\cos(Bt+C)=0$ --> $B=\frac{1}{\sqrt{LC}}\equiv\omega$

--> $Q = A\cos(\omega t - \delta)$ & $I = \frac{dQ}{dt} = -\omega A \sin(\omega t - \delta) = -I_{peak}\sin(\omega t)$

Serway/Jewett; Principles of Physics, 3/e
Figure 24.9



Capacitor --> Electric Field --> Potential Energy

Inductor --> Moving of Charges --> Kinetic Energy

The average energy stored in the capacitor (inductor) is $\frac{Q^2}{2C}$ $(\frac{1}{2}LI^2)$.

The instantaneous energy transferring in the circuit is:

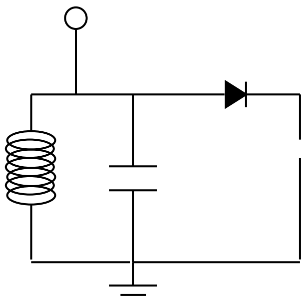
$$\frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{A^2\cos^2(\omega t - \delta)}{2C} + \frac{1}{2}L\omega^2 A^2 \sin^2(\omega t - \delta) = \frac{A^2}{2C} = \frac{Q_{peak}^2}{2C} = \frac{1}{2}LI_{peak}^2$$

What are physical pictures of Q_{peak} and I_{peak}?

Example: A 2-μF capacitor is charged to 20 V and the capacitor is then connected across a 6-μH inductor. (a) What is the frequency of oscillation? (b) What is the peak value of the current?

(a) $\omega = \frac{1}{\sqrt{LC}}$, (b) $\frac{CV^2}{2} = \frac{Q_{peak}^2}{2C} = \frac{LI_{peak}^2}{2}$

Simple AM Radio receiver:



31.6 The RLC Circuit

Kirchhoff's Rule:

$$-IR - L\frac{dI}{dt} - \frac{Q}{C} = 0$$

Power Consideration:

$$P = LI\frac{dI}{dt} + \frac{Q}{C}I = -I^2R$$

$$\rightarrow \quad L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

Compare with damped oscillation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad \rightarrow \quad x(t) = Ae^{nt}\,, \quad mn^2 + bn + k = 0 \quad \rightarrow \quad n = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

$$b^2 < 4mk \quad \rightarrow \quad x(t) = Ae^{-\frac{b}{2m}t}\cos\left(\frac{\sqrt{4mk - b^2}}{2m}t\right)$$

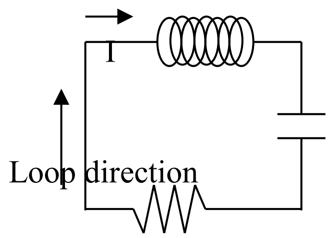
$$Q(t) = Q_{\max}e^{-\frac{R}{2L}t}\cos\left(\frac{\sqrt{4L/C - R^2}}{2L}t\right)$$

TABLE 32.1

Analogies Between Electrical and Mechanical Systems

Electric Circuit		**One-Dimensional Mechanical System**
Charge	$Q \leftrightarrow x$	Position
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving object
Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$	Damped object on a spring

RLC Circuit (Damped Oscillation)



Diff Eq: $-L\dfrac{dI}{dt}-\dfrac{Q}{C}-IR=0$

--> $L\dfrac{d^2Q}{dt^2}+R\dfrac{dQ}{dt}+\dfrac{Q}{C}=0$ (Damped Oscillations: $F=ma=-kx-bv$)

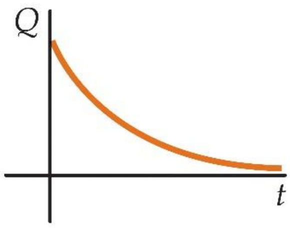
The natural frequency (no resistaor) is: $\omega_0=\dfrac{1}{\sqrt{LC}}$ ($L\dfrac{d^2Q}{dt^2}+\dfrac{Q}{C}=0$).

We guess a solution of $Q=Ae^{Bt}$ for solving the differential equation

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=0.$$

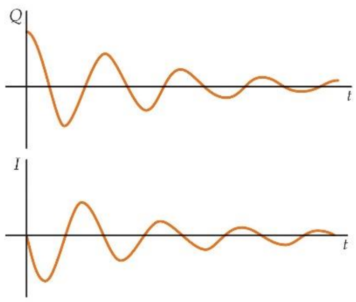
$$\left(LB^2+RB+\frac{1}{C}\right)Ae^{Bt}=0 \text{ --> } B=\frac{-R\pm\sqrt{R^2-4\frac{L}{C}}}{2L}=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

over-damped: $\left(\dfrac{R}{2L}\right)^2-\dfrac{1}{LC}>0$



under-damped: $\left(\dfrac{R}{2L}\right)^2-\dfrac{1}{LC}<0$

under-damped solution: $Q=Ae^{-\frac{R}{2L}t}e^{\pm i\sqrt{\omega_0^2-\left(\frac{R}{2L}\right)^2}\,t}$



The energy distributed in the circuit elements is:

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=0 \text{ --> } LI\frac{dI}{dt}+RI^2+\frac{Q}{C}\frac{dQ}{dt}=0$$

--> $\dfrac{d}{dt}\left(\dfrac{1}{2}LI^2+\dfrac{Q^2}{2C}\right)+I^2R=0$