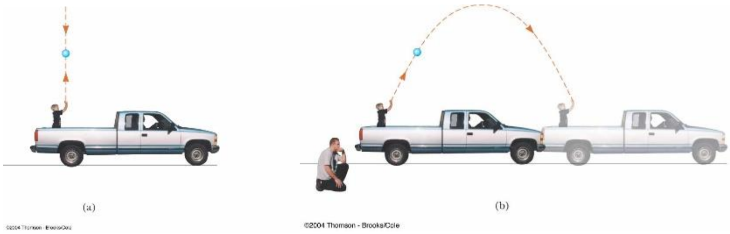
Chapter 38 Relativity

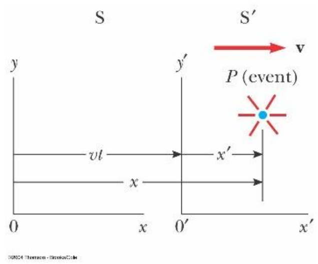
from relative motion to relativity

38.1 The Principle of Galilean Relativity

The laws of mechanics must be the same in all inertial frames of reference.



Galilean space-time transformation equations



$x' = x_{PO'} = x - vt = x_{PO} + x_{OO'}$

$y' = y$

$z' = z$

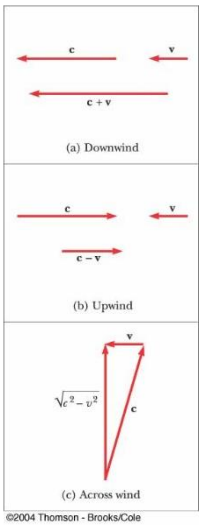
$t' = t$

Learn how to derive the "speed" in one coordinate system and the transformation to another system.

$t' = t \;\rightarrow\; dt' = dt$; $x' = x - vt \;\rightarrow\; dx' = dx - vdt$

$$\rightarrow \frac{dx'}{dt'} = \frac{dx - vdt}{dt} \quad , \quad \frac{dx'}{dt'} = v_x' = v_x - v\,,\;\; v_y' = v_y\,,\;\; v_z' = v_z \rightarrow\;\; a_x' = a_x \quad \ldots$$

The Speed of Light



Light waves moved through a medium called the ether. The speed of light was c only in a special, absolute frame at rest with respect to the ether.

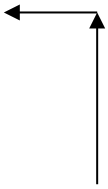
Downwind:

$c' = c + v$

Upwind:

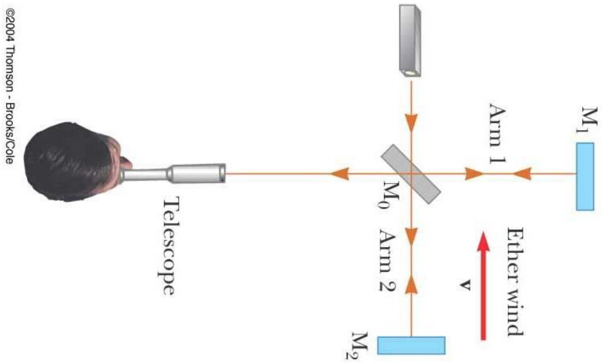
$c' = c - v$

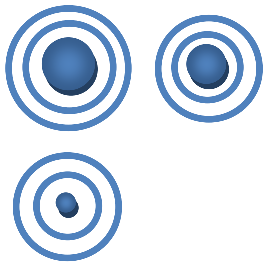
Across wind: (orthogonal to the wind flow)



$c' = \sqrt{c^2 - v^2}$

38.2 The Michelson-Morley Experiment





Phase Difference → Light Wave Interference

$$\Delta t_1 = \frac{2L}{\sqrt{c^2 - v^2}}, \quad \Delta t_2 = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2cL}{c^2 - v^2}$$

Time Difference = $\Delta t_2 - \Delta t_1 \sim \frac{2L}{c}\left[1 + \frac{v^2}{c^2} - \left(1 + \frac{v^2}{2c^2}\right)\right] = \frac{Lv^2}{c^3}$

Light 1 takes longer time than Light 2. After rotation for 90°, Light 1 takes shorter time than Light 2. The time difference between the two events is just the summation of their time differences.

Path Difference = $c(2\Delta t)$ → Phase Difference = $2\pi \times \left(2c\frac{Lv^2}{c^3}\right)/\lambda$

Shift of fringe pattern: $\left(2c\frac{Lv^2}{c^3}\right)/\lambda$

38.3 Einstein's Principle of Relativity

1. The Principle of Relativity: The law of physics must be the same in all inertial reference frames.

All the laws of physics – those dealing with mechanics, electricity and magnetism, optics, thermodynamics, and so on – are the same in all reference frames moving with constant velocity relative to one another.

From an experimental point of view, it means that any kind of experiment performed in a laboratory at rest must give the same result when performed in a laboratory moving at a **constant velocity** with respect to the first one.

2. The Constancy of The Speed of Light: The speed of light in vacuum has the same value in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

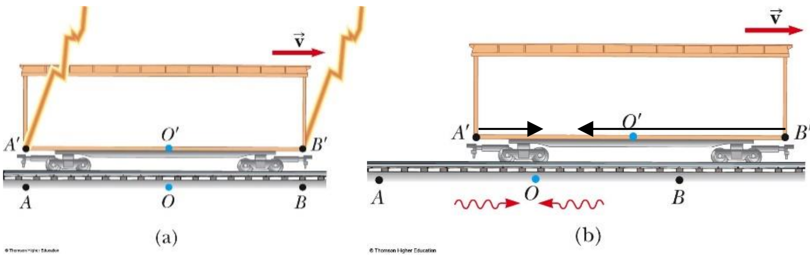
The motion of the Earth does not influence the fringe pattern observed in the Michelson-Morley experiment, and a null result should be expected.

38.4 Consequences of Special Theory of Relativity

Here we restrict our discussion to the concepts of simultaneity, time intervals, and lengths.

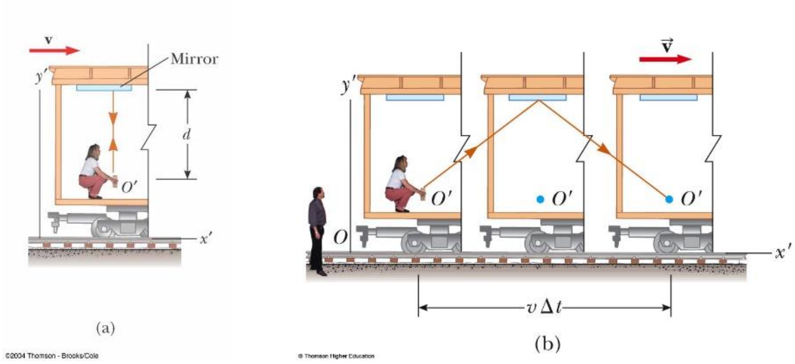
1. Simultaneity

Definition of Simultaneity: two lights start at the same time and reach the same position at the same later time.
Two events that are simultaneous in one reference frame are in general not simultaneous in a second frame moving relative to the first.



The signal from B' has already swept past O'. For O', the light from B' emits earlier.

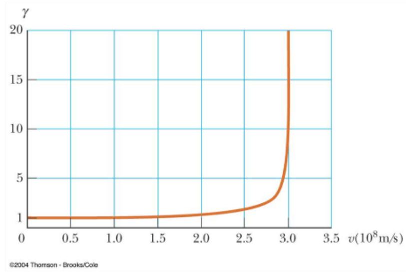
2. Time Dilation



$$\Delta t' = \frac{2d}{c}, \quad \frac{\Delta t}{2} = \frac{\sqrt{d^2 + \left(v\dfrac{\Delta t}{2}\right)^2}}{c}$$

$$\rightarrow \quad \Delta t = 2\frac{d}{\sqrt{c^2 - v^2}} = 2\frac{d}{c}\frac{1}{\sqrt{1-(v/c)^2}}$$

$$\rightarrow \quad \Delta t = \frac{\Delta t'}{\sqrt{1-(v/c)^2}} = \gamma \Delta t', \quad \gamma = \frac{1}{\sqrt{1-(v/c)^2}} > 1$$



The time interval of the event measured at rest is the shortest.
The time interval $\Delta t'$ is called the **proper time interval**.

The proper time interval is the time interval between two **events** measured by an **observer** who sees that the events occur at the same point in space without any relative motion.

If a clock is moving with respect to you, the time interval between ticks of the moving clock is observed to be longer than the time interval between ticks of an identical clock in your reference frame.

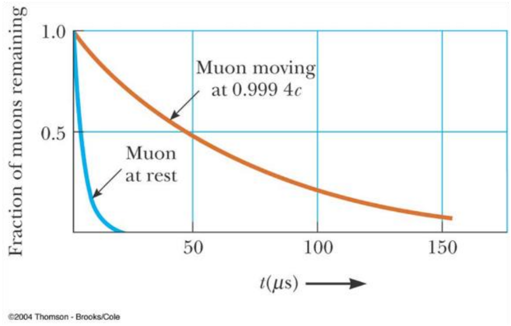
Example: The period of a pendulum is measured to be 3.00 s in the reference frame of the pendulum. What is the period when measured by an observer moving at a speed of 0.960c relative to the pendulum?

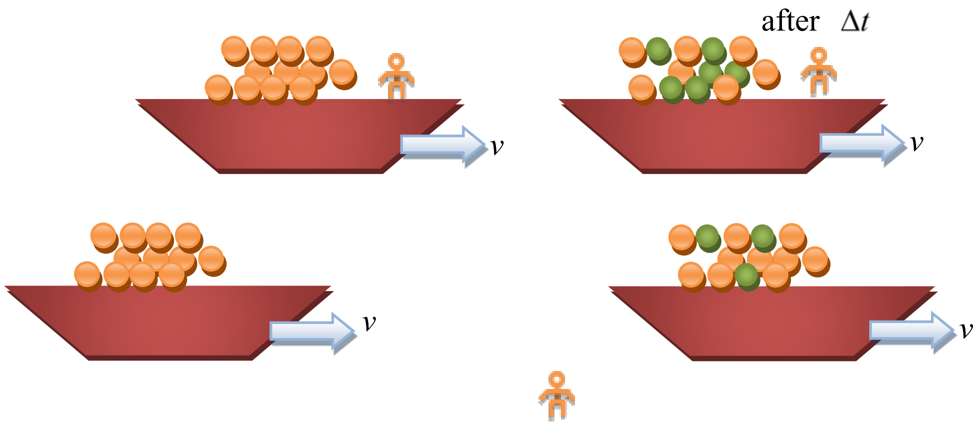
Instead of observer moving at 0.960c, we can take the equivalent point of view that the observer is at rest and the pendulum is moving at 0.960c past the stationary observer.

$$\Delta t = \gamma \Delta t' \rightarrow \Delta t = \frac{3}{\sqrt{1-(0.96)^2}} = 10.7 \quad (\mathrm{s})$$

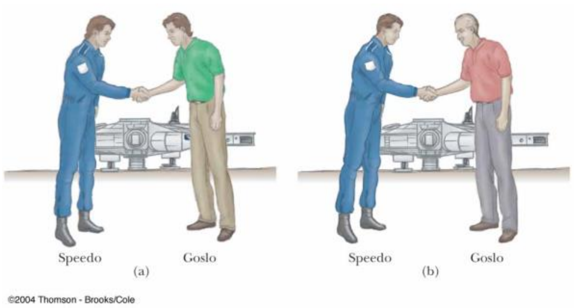
The flying clocks lost 59±10 ns during the eastward trip and gained 273±7 ns during the westward trip.

The earth rotates to the westward. If the flying clocks move to the eastward trip, the clocks are more close to the rest earth without rotation and the measured time is shorter. On the contrary, the clocks move faster than the earth’s rotation to the westward trip thus the measured time is longer.





3. The Twin Paradox



Speedo: spacecraft reaches a speed of 0.95c, travels to Planet X located 20 lightyears away.

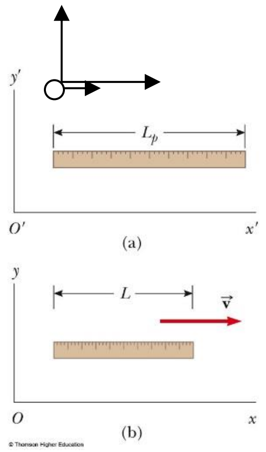
Lightyear: $c \times 365 \times 24 \times 60 \times 60 = 9.46 \times 10^{15}$ (m)

Goslo: inertial frame

Only Goslo who is always in a single inertial frame can make correct predictions based on special relativity.

Upon Speedo's return, Goslo has aged 42 years. Goslo finds that Speedo ages only

$\sqrt{1-v^2/c^2}\times 42 = 13$.



4. Length Contraction

The proper length $L = L_p$ of an object is the length measured by someone **at rest relative to the object.**

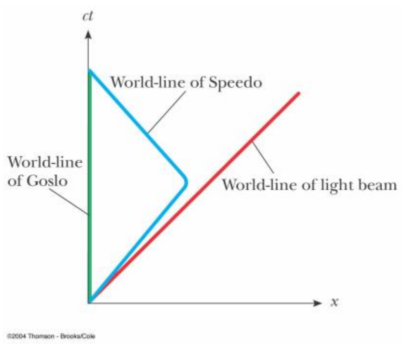
O' at rest, O move with a speed v

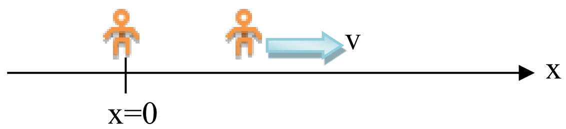
In O', the observer measures the time period for O to complete the measurement as $\Delta t'$, while he measures the proper length L at rest. Observer O': $L = v\Delta t'$

On the contrary, the observer in O measures the proper time Δt whereas he measures the length L' in motion. Observer O: $L' = v\Delta t$

$\rightarrow \frac{L'}{L} = \frac{\Delta t}{\Delta t'} = \frac{1}{\gamma} \rightarrow \gamma L' = L \rightarrow L_p = \gamma L'$

5. Space-Time Graphs





All possible future events lie above the x-axis and between the red lines.

Example: A Voyage to Sirius

An astronaut takes a trip to Sirius, which is located a distance of 8 lightyears from the Earth. The astronaut measures the time of the one-way journey to be 6 years. If the spaceship moves at a constant speed of 0.8c, how can the 8-ly distance be reconcile with the 6-year trip time measured by the astronaut?

Length contraction: $8\sqrt{1-v^2/c^2} = 8\sqrt{1-0.64} = 4.8$ lightyears

Travel time: $4.8c/0.8c = 6$ years

6. The Relativistic Doppler Effect

If a light source and a observe approach each other with a relative speed v

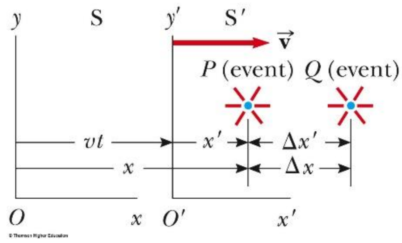
$$f_{obs} = \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}} f_{source}$$

38.5 The Lorentz Transformation Equation

Now we look for a transformation like this form:

$$x' = \gamma(x - vt)$$

$$x = \gamma(x' + vt')$$



Given a light emitting from $x = x' = 0$ at $t = t' = 0$, the event will give the relation at later time:

$$ct' = \gamma(ct - vt)$$

$$ct = \gamma(ct' + vt')$$

→ $ct = \gamma(c+v)t' = \gamma(c+v)\frac{\gamma}{c}(c-v)t$

→ $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$

$$x' = \gamma(x - vt), \quad x = \gamma(x' + vt')$$

From $x = \gamma(x' + vt')$ → $t' = \frac{1}{\gamma v}(x - \gamma x') = \frac{1}{\gamma v}\left(x - \gamma^2(x - vt)\right) = \frac{1}{\gamma v}\left(\gamma^2 vt + (1-\gamma^2)x\right)$

$$t' = \frac{1}{\gamma v}\left(\gamma^2 vt + (1-\gamma^2)x\right) = \frac{1}{\gamma v}\left(\gamma^2 vt + \left(\frac{-v^2/c^2}{1-v^2/c^2}\right)x\right)$$

$$t' = \frac{1}{\gamma v}\left(\gamma^2 vt - \gamma^2 \frac{v^2}{c^2} x\right) = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Displacement Transformation:

$\Delta x' = x_2' - x_1'$, $\Delta x = x_2 - x_1$, $\Delta t' = t_2' - t_1'$, $\Delta t = t_2 - t_1$

→ $\Delta x' = \gamma(\Delta x - v\Delta t)$, $\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$

Use $x' = \gamma(x - vt)$ and $t' = \gamma\left(t - \frac{vx}{c^2}\right)$, to derive $x = \gamma(x' + vt')$.

Example: Simultaneity and Time Dilation Revisited

(a) Imagine two events that are simultaneous and separated in space such that $\Delta t' = 0$ and $\Delta x' \neq 0$ **according to an observer O'** who is moving with speed v respect to O.

$$\Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) = \gamma\frac{v}{c^2}\Delta x'$$

Simultaneous in O' → not simultaneous in O

Simultaneous in O → not simultaneous in O'

(b) A moving clock is measured to run more slowly than a clock that is at rest with respect to an observer. → $\Delta x' = 0$

O' is moving with respect to O

$$\Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) = \gamma\Delta t'$$

38.6 The Lorentz Velocity Transformation Equation

$$x' = \gamma(x - vt),\quad y' = y,\quad z' = z,\quad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$u_x' = \frac{dx'}{dt'} = \frac{dx - vdt}{dt - \frac{v}{c^2}dx} = \frac{u_x - v}{1 - \frac{v}{c^2}u_x} \rightarrow u_x = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}$$

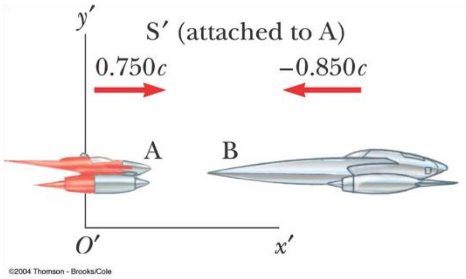
$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

$$x = \gamma(x' + vt'),\quad y' = y,\quad z' = z,\quad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

Please derive the transform from u_x' to u_x.

Example: Relative Velocity of Two Spacecraft

Two spacecraft A and B are moving in opposite direction. An observer on the Earth measures the speed of spacecraft A to be 0.750c and the speed of spacecraft B to be 0.850c. Find the velocity of spacecraft B as observed by the crew on spacecraft A.



B to O’ → $u'_x = -0.850c$

O stick on A thus O move 0.750c to the right w.r.t. O’

$v_{oo'} = 0.75c$ → $v_{o'o} = -0.75c$ $\quad u_x = \dfrac{u'_x + v}{1 + \dfrac{v}{c^2} u'_x}$

→ $u_x = \dfrac{-0.850c - 0.750c}{1 - \dfrac{0.750c}{c^2}(-0.850c)} = -0.977c$ with respect to O and also to spacecraft A

38.7 Relativistic Linear Momentum

Classical Definition: $p = mu$

Here $u = \dfrac{\Delta x}{\Delta t}$ from the observer at rest.

The moving particle has its proper time $\Delta\tau = \dfrac{\Delta t}{\gamma}$

Thus the momentum of the particle shall be

$$p = mu = m\frac{\Delta x}{\Delta \tau} = m\frac{\Delta x}{\Delta t}\gamma = m\frac{\Delta x}{\sqrt{1-u^2/c^2}\,\Delta t} = \frac{mu}{\sqrt{1-u^2/c^2}}$$

Force: $F = \dfrac{dp}{dt}$

38.8 Relativistic Energy

The acting force gives a work that transforms to kinetic energy of an object.

$$\vec{p}(\vec{u}) = \frac{m\vec{u}}{\sqrt{1-(u/c)^2}}$$

$$K=\int_{x1}^{x2}Fdx=\int_0^t\frac{dp}{dt}\frac{dx}{dt}dt=\int_0^t\frac{dp}{dt}udt=\int_0^t\frac{dp}{du}\frac{du}{dt}udt=\int_0^v\frac{dp}{du}udu$$

$$\vec{p}=\frac{m\vec{u}}{\sqrt{1-(u/c)^2}} \rightarrow \begin{aligned}\frac{dp}{du}&=\frac{m}{\sqrt{1-(u/c)^2}}-\frac{1}{2}\frac{mu}{\left(1-(u/c)^2\right)^{3/2}}\left(-2\frac{u}{c^2}\right)\\&=m\left(\frac{1}{\left(1-(u/c)^2\right)^{3/2}}\right)\end{aligned}$$

$$K=m\int_0^{u'}\frac{u}{\left(1-u^2/c^2\right)^{3/2}}du=m\int_0^{u'}d\left(c^2\frac{1}{\sqrt{1-u^2/c^2}}\right) \rightarrow K=E-mc^2$$

$$=\frac{mc^2}{\sqrt{1-u'^2/c^2}}-mc^2$$

$E=K+mc^2=\gamma mc^2 \rightarrow E=\gamma mc^2$ when $u=0$, $E=mc^2$

$$p=\gamma mv \rightarrow p^2-p^2\frac{v^2}{c^2}=m^2v^2 \rightarrow v^2=\frac{p^2}{m^2+p^2/c^2}$$

$$\rightarrow E^2-E^2\frac{v^2}{c^2}=m^2c^4 \rightarrow E^2-\frac{E^2}{c^2}\frac{p^2}{m^2+p^2/c^2}=m^2c^4$$

$$E^2-m^2c^4=\frac{p^2E^2}{m^2c^2+p^2} \rightarrow E^2m^2c^2-m^4c^6+p^2E^2-p^2m^2c^4=p^2E^2$$

$$\rightarrow E^2-m^2c^4-p^2c^2=0 \rightarrow E^2=p^2c^2+m^2c^4$$

Photon has zero mass → $E=pc$ (the particle reaching the light speed shall have zero mass)

Example: (a) Find the rest energy of a proton in the unit of eV. (b) If the total energy of a proton is three times of its rest energy. What is the speed of the proton? (c) Determine the kinetic energy of the proton in the unit of eV. (d) What is the proton's momentum?

(a) $m_p=1.673\times10^{-27}$ kg, $E=m_pc^2=1.5057\times10^{-10}(J)=9.4\times10^8(eV)$

(b) $E=\gamma m_pc^2=3m_pc^2 \rightarrow \left(\frac{v}{c}\right)^2=1-\frac{1}{9}=\frac{8}{9} \rightarrow v=\frac{2\sqrt{2}}{3}c=2.83\times10^8$ m/s

(c) $K=E-m_pc^2=2m_pc^2 \rightarrow K=1.88\times10^9$ (eV)

(d) $p=\gamma mu=3\times1.673\times10^{-27}\times2.83\times10^8=1.42\times10^{-18}$ (kg m/s)

38.9 Mass and Energy

nuclear and elementary-particle interactions

→ **we cannot use the principle of conservation of energy**

conventional nuclear reactor: the uranium nucleus undergoes fission → resulting in several lighter fragments of atoms

the total mass of the fragment atoms is less than the uranium atom by an amount Δm, thus giving an energy of Δmc^2 → mass-energy transformation

fusion reaction: two deuterium atoms combine to form one helium atom, resulting in the reduction of total mass by an amount $\Delta m = 4.25\times10^{-29}$ kg, thus giving an energy of 23.9 MeV

Example: The ^{216}Po nucleus is unstable and exhibits radioactivity. It decays to ^{212}Pb by emitting an alpha particle (^{4}He). The relevant mass are m(^{216}Po)=216.001 915 u, m(^{4}He)=4.002 603, and m(^{212}Pb)=211.991 898 u. (a) Find the mass change. (b) Find the energy from the fission reaction.

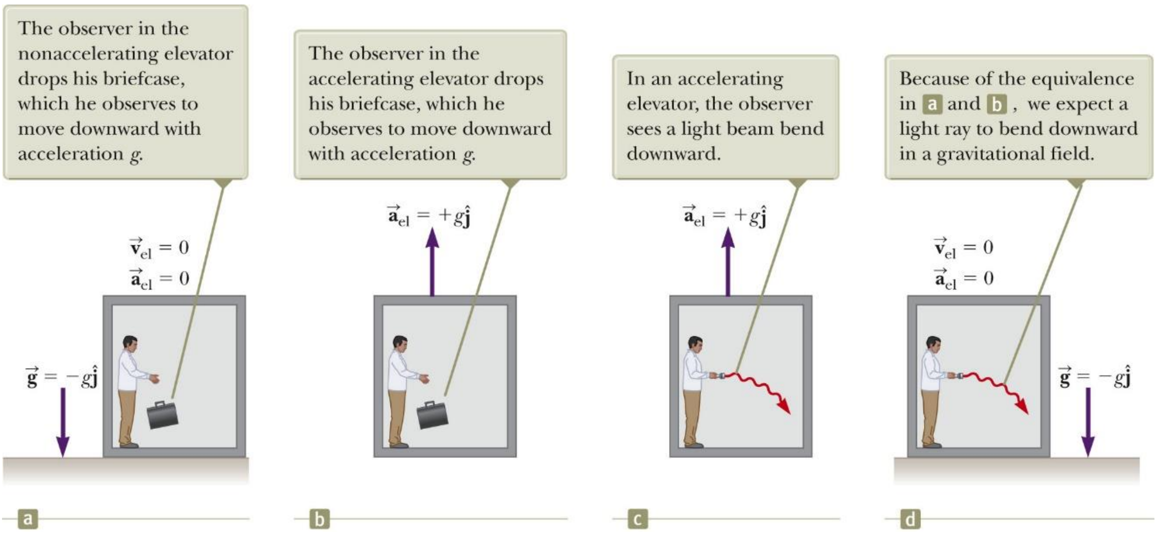
(a) $216.001915 - 211.991898 - 4.002603 = 0.007414u = 1.23\times10^{-29}$ kg

(b) $1.23\times10^{-29}\times\left(3\times10^{8}\right)^2 = 1.11\times10^{-12}$ J

38.10 The General Theory of Relativity

A beam of light should also be bent downward by a gravitational field

1. All the laws of nature have the same form for observers in any frame of reference, whether accelerated or not.
2. In the vicinity of any point, a gravitational field is equivalent to an accelerated frame of reference in gravity-free space (the principle of equivalence).



Time is altered by gravity. A clock in the presence of gravity runs slower than one located where gravity is negligible.

The frequency of radiation emitted by atoms in the presence of a strong gravitational field is redshifted to a lower frequency. The time near the strong gravitational field is shorter than that in the presence of a weak field, thus the frequency near the strong gravitational field is higher. When the frequency is observed near the weak field space, the frequency is redshifted.

The presence of a mass causes a curvature of space-time in the vicinity of the mass and curvature dictates the space-time path that all freely moving objects must follow.