



Physics I Lecture01 - Mathematics-Part I

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1. QUADRATIC EQUATION

Example: When a driver spots a running motorcycle moving with a constant speed (10 m/s) a distance of 5 m ahead of his car, he starts to accelerate at a 6 m/s^2 . When will the driver catch up with the motorcycle?

$$S_M(t) = 5 + 10t$$

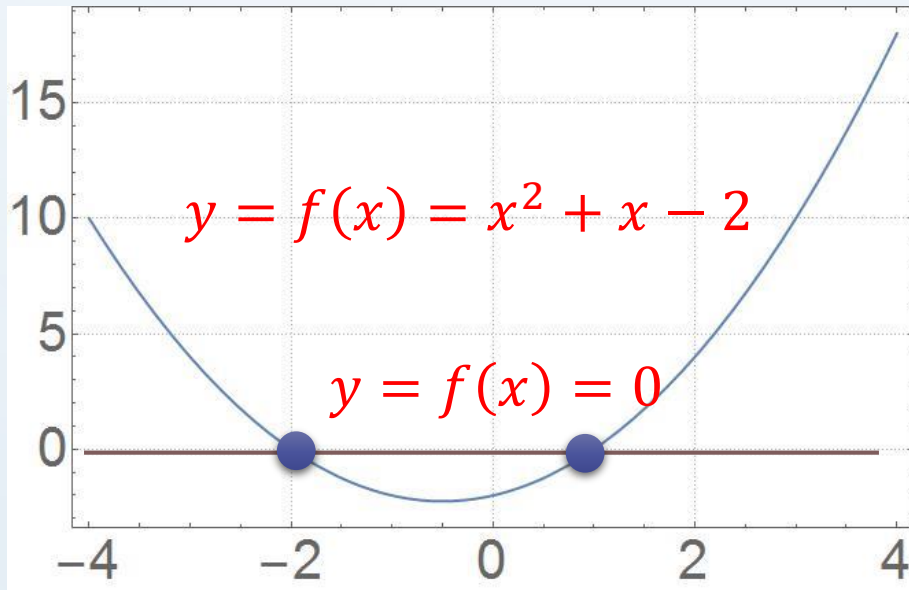
$$S_C(t) = \frac{6}{2}t^2$$

Assume that the driver catch up with the motorcycle at $t = t_1$

$$S_M(t_1) = S_C(t_1) \rightarrow 5 + 10t_1 = 3t_1^2$$

$$3t_1^2 - 10t_1 - 5 = 0 \rightarrow t_1 = \frac{10 \pm \sqrt{10^2 + 4 \times 5 \times 3}}{2 \times 3} = -0.44 \text{ or } 3.8 \text{ (s)}$$

1. QUADRATIC EQUATION



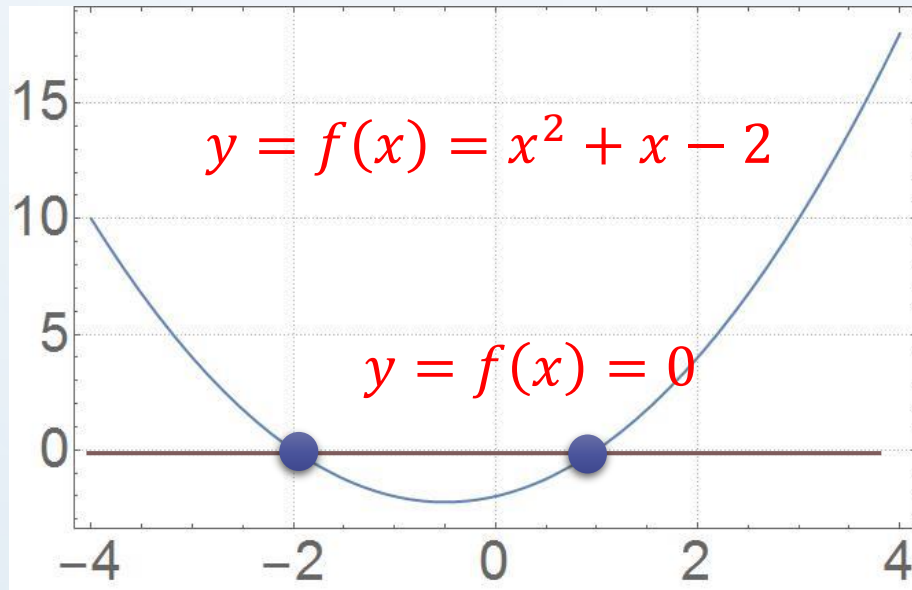
$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

1. QUADRATIC EQUATION



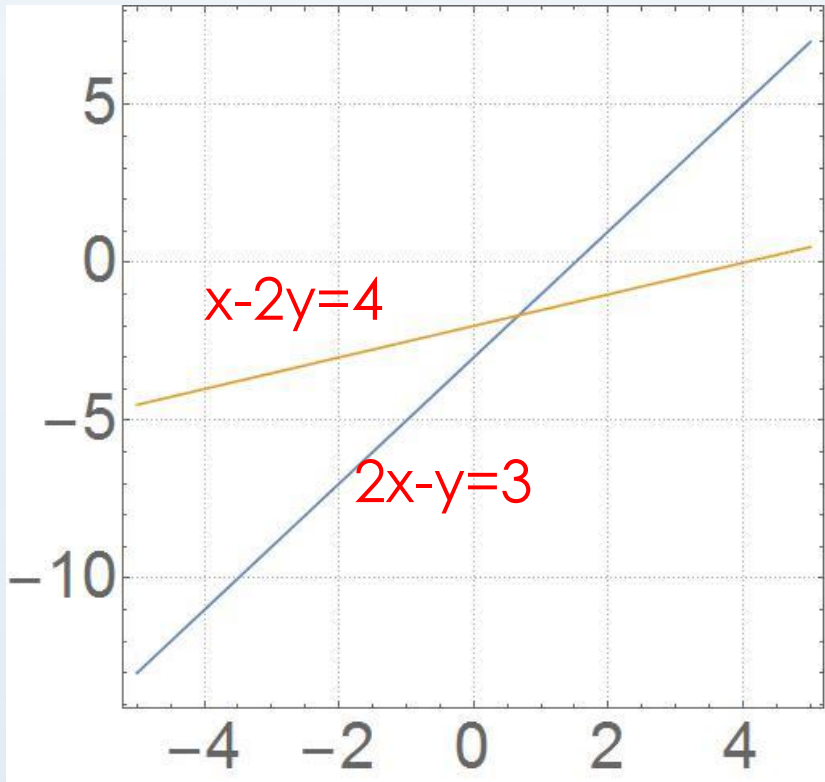
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{(2a)^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{(2a)^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. LINEAR EQUATIONS



$$- \quad a_1x + b_1y = c_1 \quad \times a_2$$

$$+ \quad a_2x + b_2y = c_2 \quad \times a_1$$

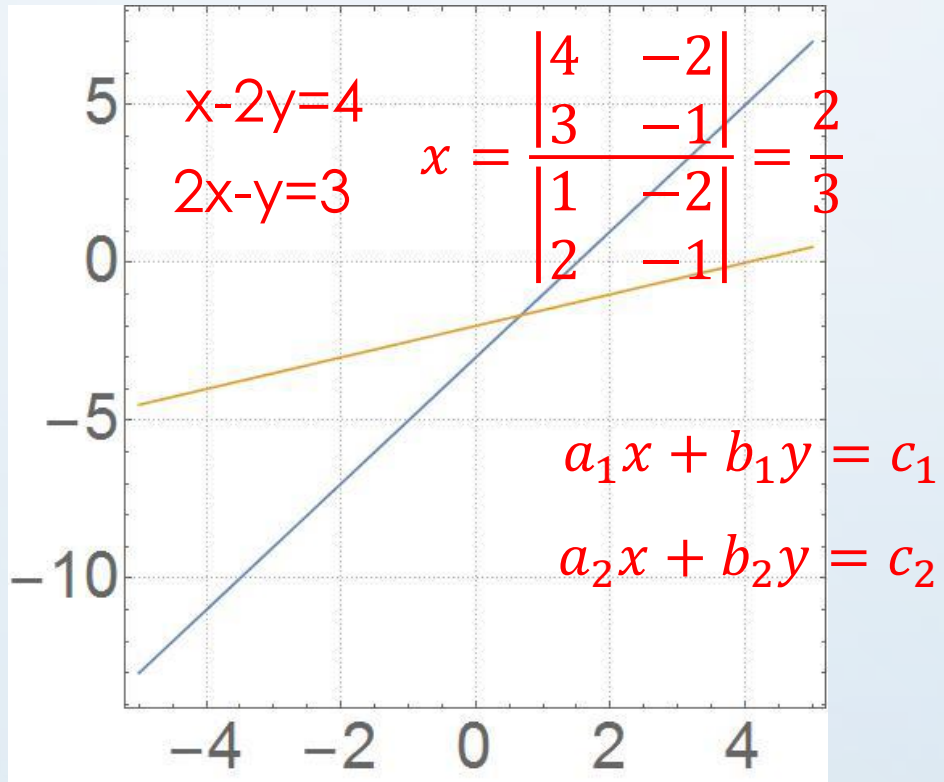
$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

2. LINEAR EQUATIONS

Cramer's Rule



$$a_1c_2 - a_2c_1 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

2. LINEAR EQUATIONS

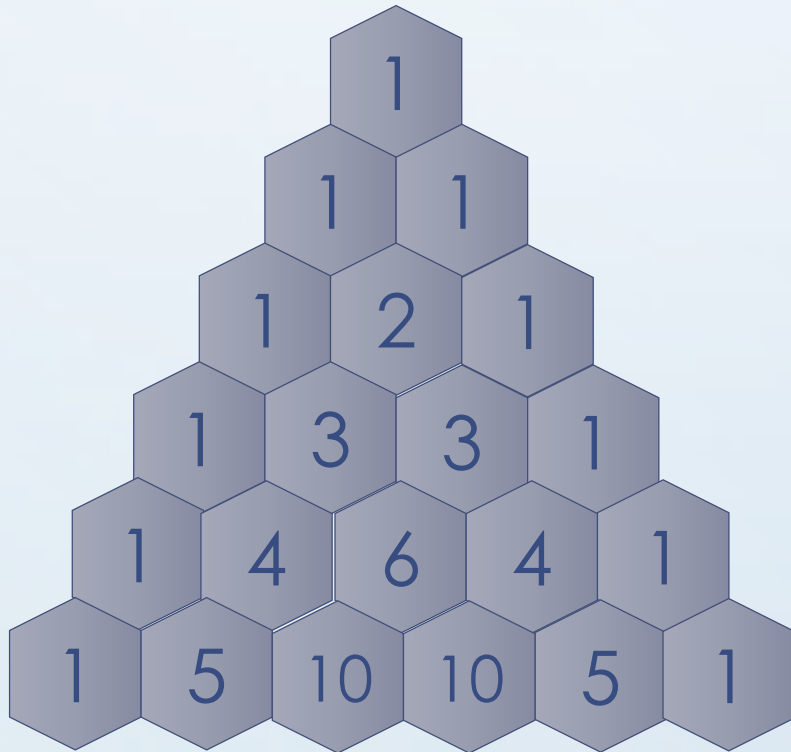


$$\begin{aligned}a_1x + b_1y + c_1z &= d_1 \\a_2x + b_2y + c_2z &= d_2 \\a_3x + b_3y + c_3z &= d_3\end{aligned}$$

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

3. BINOMIAL SERIES

Pascal's triangle:



$$(1 + x)^2 = (1 + x)(1 + x)$$

$$(1 + x)^2 = 1 + 2x + x^2$$

$$(1 + x)^2 = C_0^2 + C_1^2 x + C_2^2 x^2$$

$$(1 + x)^n = C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n$$

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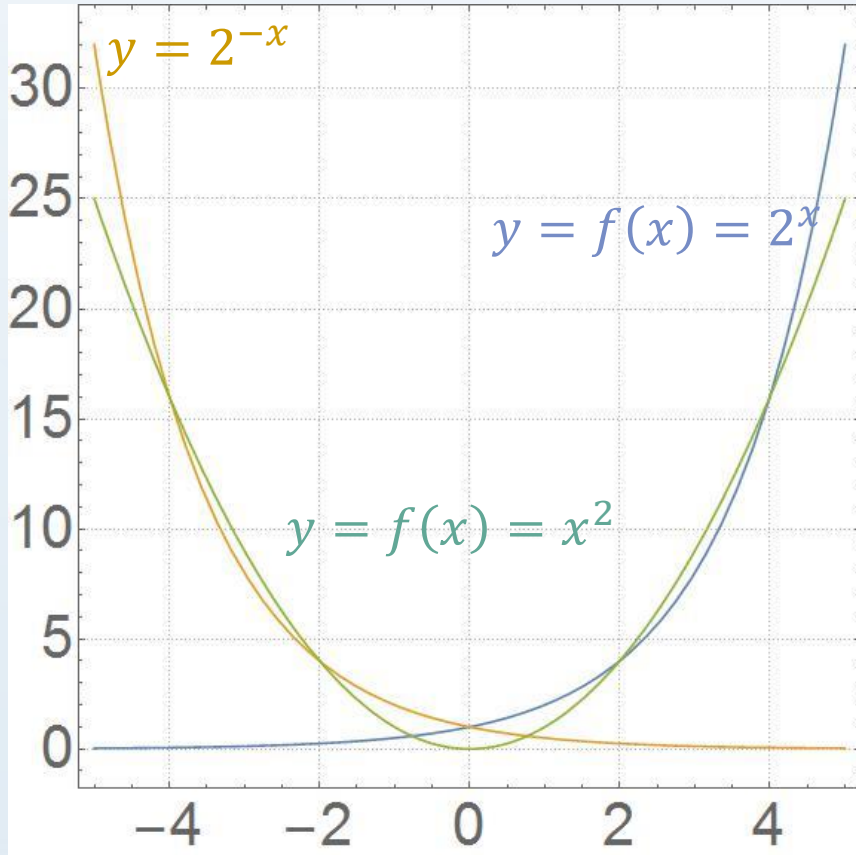
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4. POWER, EXPONENT



$$a^m = a^{m-n+n} \Rightarrow a^x \text{ such as } 2^x, 2^{-x}$$

$$a^m = a^{m-n} \times a^n$$

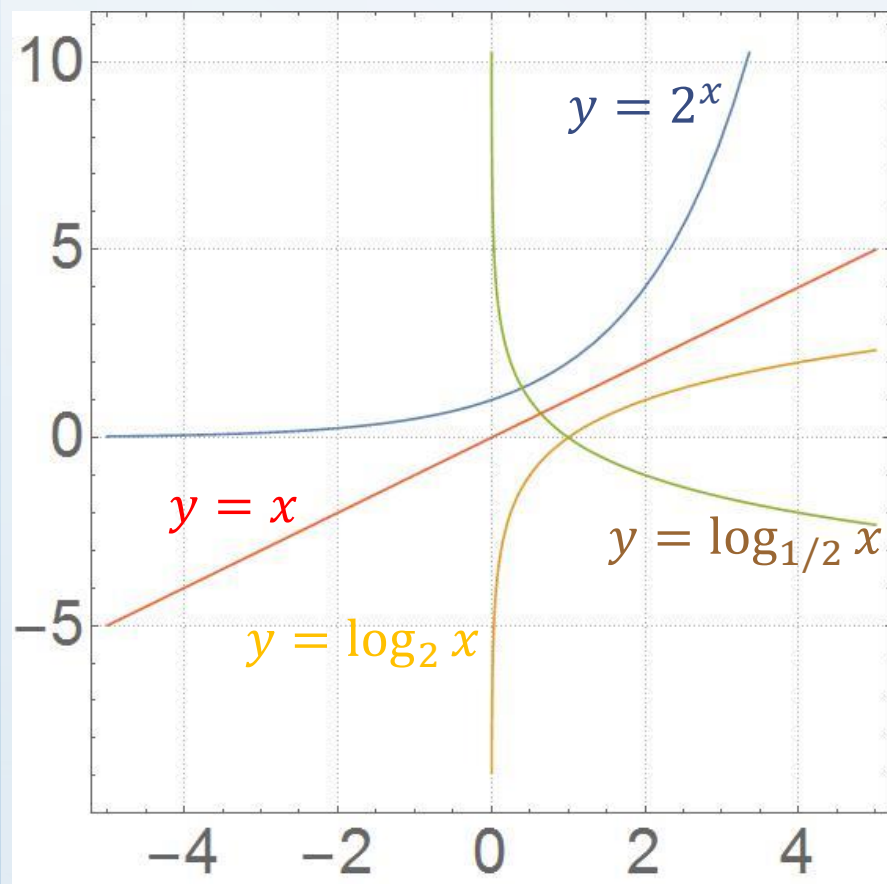
$$\frac{a^m}{a^n} = a^{m-n}$$

$$m = 0, \frac{1}{a^n} = a^{-n}$$

$$A \times A = a = a^{\frac{1}{2} + \frac{1}{2}}$$

$$A = a^{\frac{1}{2}} = \sqrt{a}$$

5. LOGARITHMS



$$y = a^x$$

$$x = \log_a(y)$$

$$100 = 10^2$$

$$\log_{10}(100) = 2$$

$$\log_{10}(1) = 0$$

$$y_1 = a^x, y_2 = a^y$$

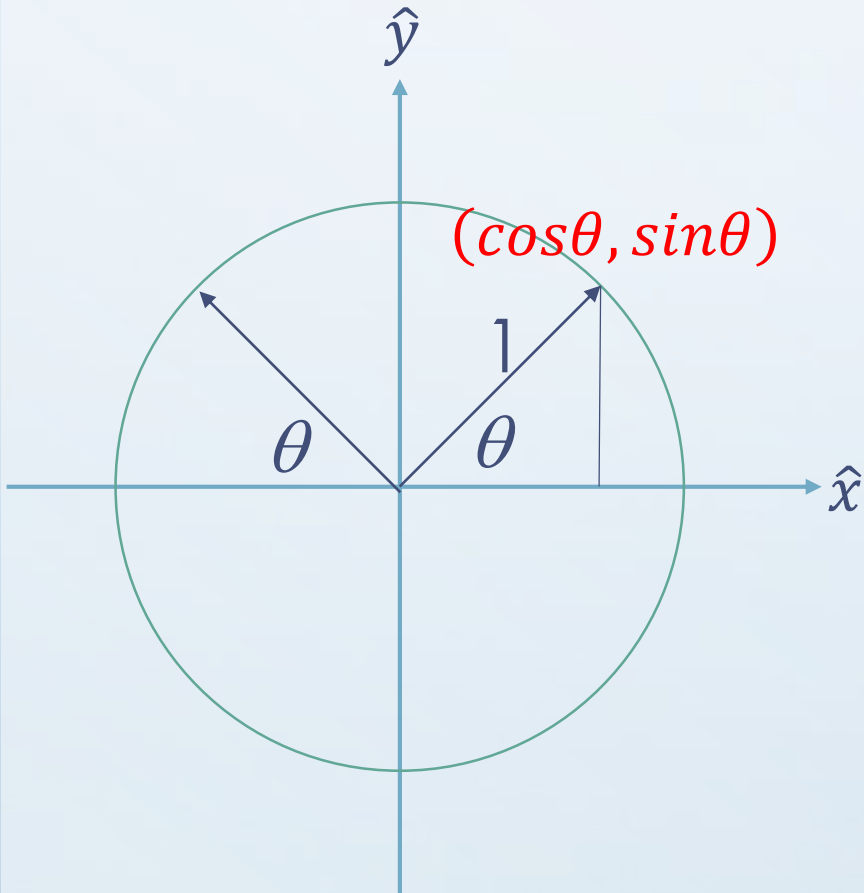
$$y_1 y_2 = a^x a^y = a^{x+y}$$

$$\begin{aligned} \log(y_1 y_2) &= x + y \\ &= \log(y_1) + \log(y_2) \end{aligned}$$

$$e = 2.718281828459045$$

$$\log_e y = \ln(y)$$

6. TRIGONOMETRIC FUNCTIONS



addition formula

$$\begin{aligned} \sin(A \pm B) \\ = \sin A \cos B \pm \cos A \sin B \end{aligned}$$

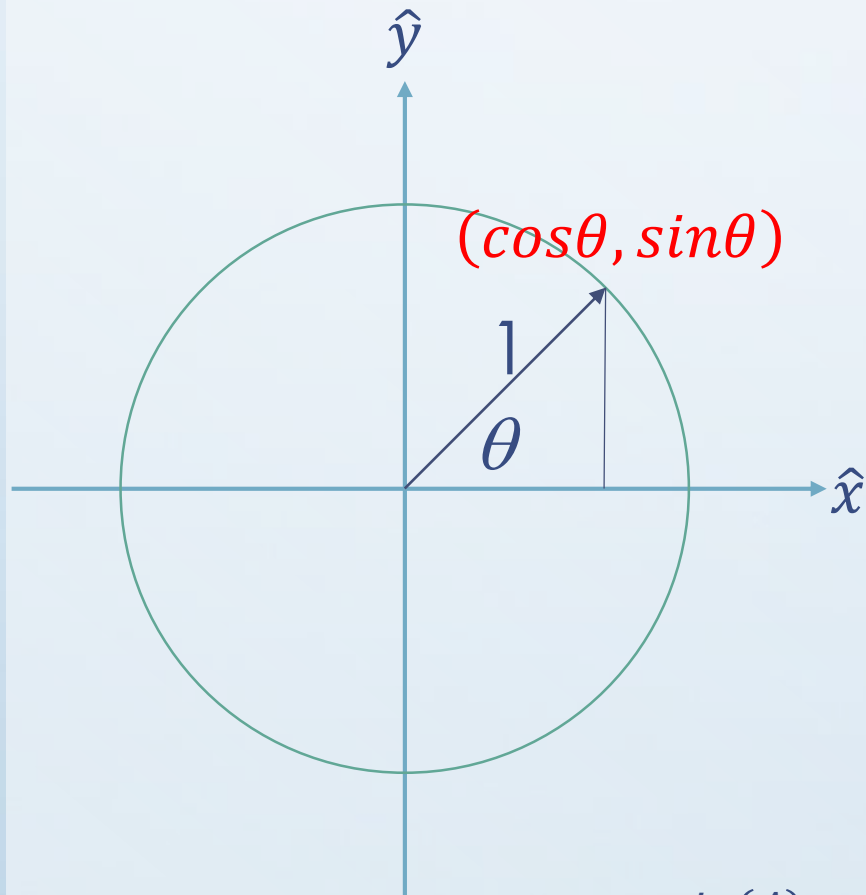
$$\begin{aligned} \cos(A \pm B) \\ = \cos A \cos B \mp \sin A \sin B \end{aligned}$$

$$\sin(\pi - \theta) = \sin \theta$$

$$\cos(\pi - \theta) = -\cos \theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

6. TRIGONOMETRIC FUNCTIONS



$$\sin(A) = \sin\left(\frac{A+B}{2} + \frac{A-B}{2}\right)$$

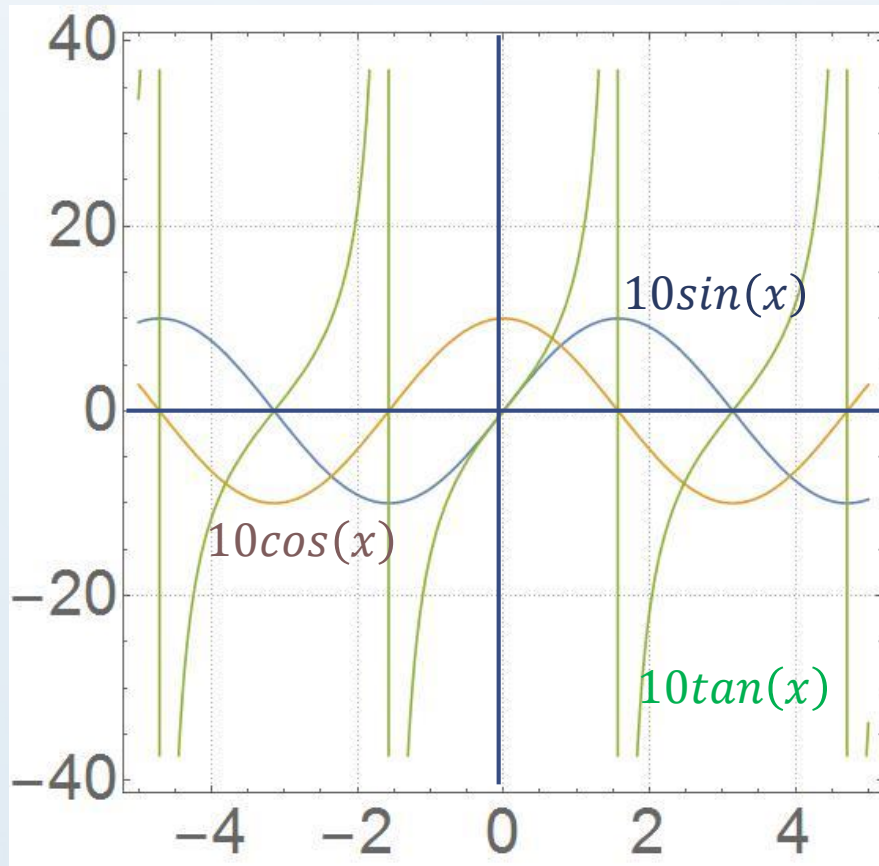
$$\sin(A) = \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin(B) = \sin\left(\frac{A+B}{2} - \frac{A-B}{2}\right)$$

$$\sin(B) = \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

6. TRIGONONAL FUNCTIONS



$$y = f(x) = 10\sin(x)$$

$$y = f(x) = 10\cos(x)$$

$$y = f(x) = 10\tan(x)$$

APPLICATION OF SINUSOIDAL FUNCTIONS

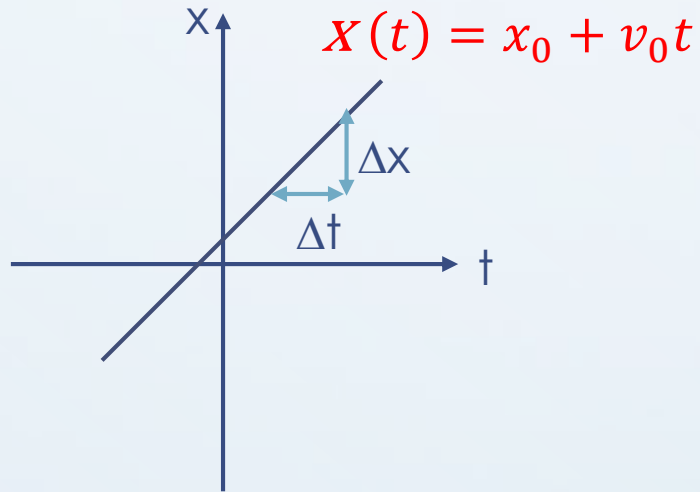


Lissajous curve

$$x(t) = A \sin(Ct + \delta)$$

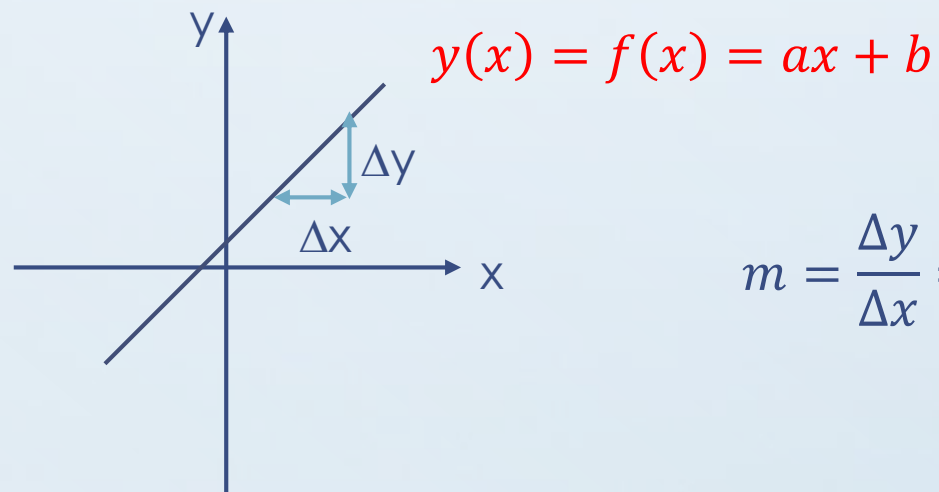
$$y(t) = B \sin(Dt)$$

7. DIFFERENTIAL CALCULUS – CONSTANT VELOCITY



constant velocity

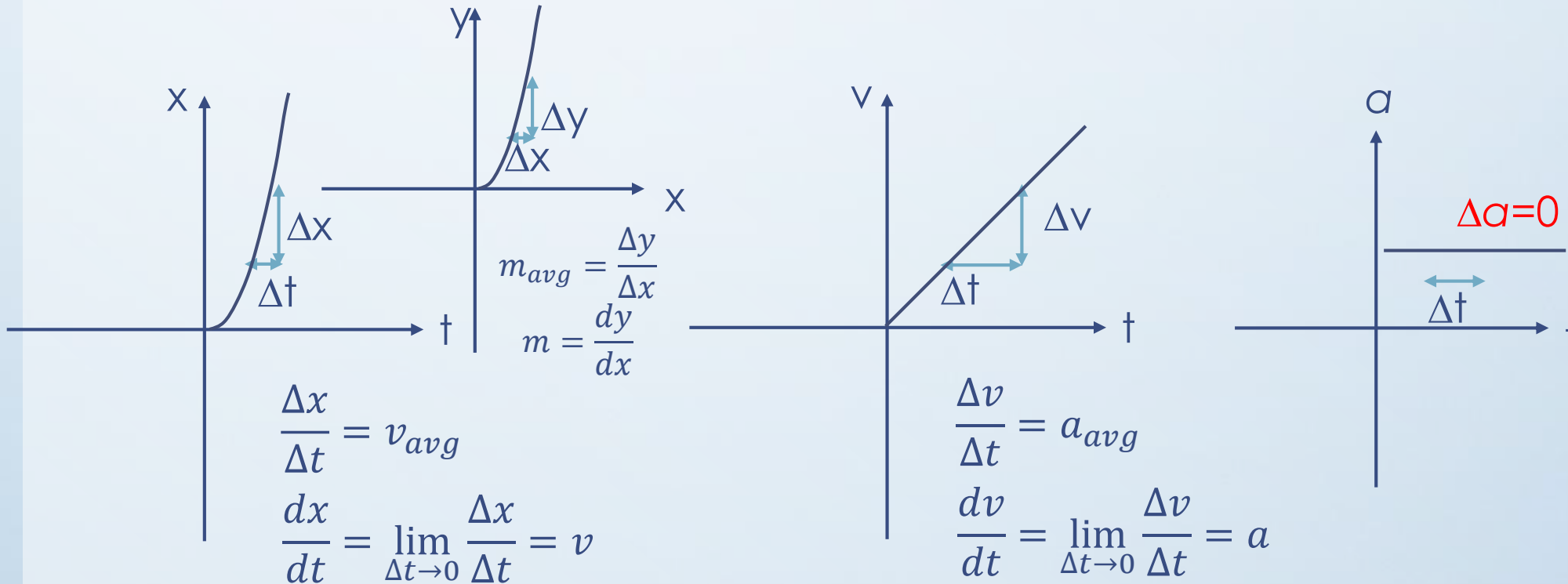
$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$
$$= \frac{x(t_2) - x(t_1)}{t_2 - t_1} = v_0$$



$$m = \frac{\Delta y}{\Delta x} = \frac{y(x_2) - y(x_1)}{x_2 - x_1}$$

7. DIFFERENTIAL CALCULUS – CONSTANT ACCELERATION

constant acceleration – freely falling object



Differential Calculation: $\frac{dx(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{x(t+\Delta t) - x(t)}{t+\Delta t - t}$

7. DIFFERENTIAL CALCULUS – DEFINITION OF DIFFERENTIATION

Differential Calculation: $\frac{dx(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{x(t+\Delta t) - x(t)}{t+\Delta t - t}$

For example, $x(t) = t^2$,

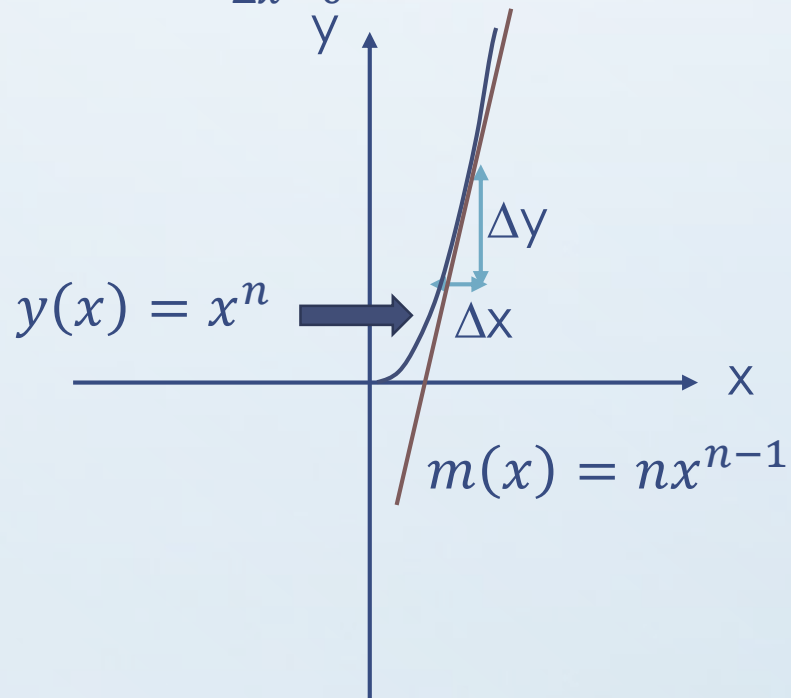
$$\frac{dx(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{(t + \Delta t)^2 - t^2}{\Delta t} = 2t$$

$$\frac{d(x^n)}{dx} = nx^{n-1} \quad \frac{d(\sin(x))}{dx} = \cos(x) \quad \frac{d(\cos(x))}{dx} = -\sin(x)$$

$$\frac{d(e^x)}{dx} = e^x \quad \frac{d(\ln(x))}{dx} = \frac{1}{x}$$

7. DIFFERENTIAL CALCULUS – DEFINITION OF DIFFERENTIATION

$$\begin{aligned}\frac{d}{dx}(x^n) &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^n + C_1^n x^{n-1} \Delta x + C_2^n x^{n-2} (\Delta x)^2 + \dots - x^n}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (C_1^n x^{n-1} + C_2^n x^{n-2} \Delta x + \dots) = nx^{n-1}\end{aligned}$$



$$d(x^n) = m(x)dx = nx^{n-1}dx$$

$$dy = \frac{dy}{dx} dx = m(x)dx$$

7. DIFFERENTIAL CALCULUS – CHAIN RULE

rules of differentiation

$$\frac{d}{dx} [f(x)g(x)] = \frac{df(x)}{dx} g(x) + f(x) \frac{dg(x)}{dx}$$

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{d}{dx} [f(x)g^{-1}(x)] = \frac{df(x)}{dx} g^{-1}(x) + f(x) \frac{dg^{-1}(x)}{dx}$$

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{df(x)}{dx} g^{-1}(x) + f(x) \left(-g^{-2}(x) \frac{dg(x)}{dx} \right)$$

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{\left(\frac{df(x)}{dx} g(x) - f(x) \frac{dg(x)}{dx} \right)}{g^2(x)}$$

$$\frac{d}{dx} [f(g(x))] = \frac{df(g)}{dg} \frac{dg(x)}{dx} \quad \text{chain rule}$$

7. DIFFERENTIAL CALCULUS – CHAIN RULE

All calculations are based on the knowledge you already learned!

$$\frac{d(\sin(x))}{dx} = \cos(x) \quad \& \quad \frac{dx^3}{dx} = 3x^2 \quad \rightarrow \quad \frac{d \sin(x^3)}{dx} = ?$$

$$\text{Let } f(g) = \sin(g), \quad g(x) = x^3 \quad \frac{df(g(x))}{dx} = \frac{df(g)}{dg} \frac{dg(x)}{dx}$$

$$\frac{d \sin(x^3)}{dx} = \frac{d \sin(x^3)}{d(x^3)} \frac{dx^3}{dx} = \cos(x^3) 3x^2$$

$$\frac{d(\cos(x^2 + 2x) + x^3)}{dx} = \left(\frac{d}{dx} \right) (\cos(x^2 + 2x) + x^3) = \frac{d \cos(x^2 + 2x)}{dx} + \frac{dx^3}{dx}$$

$$= \frac{d \cos(x^2 + 2x)}{d(x^2 + 2x)} \frac{d(x^2 + 2x)}{dx} + 3x^2 = -\sin(x^2 + 2x) (2x + 2) + 3x^2$$

8. PARTIAL DIFFERENTIAL CALCULUS – ORTHOGONAL COORDINATES

Vector in space to demonstrate the concept

Linearly Dependent: $a\vec{A} + b\vec{B} = 0, a \neq 0, b \neq 0$

Linearly Independent: $a\vec{A} + b\vec{B} = 0, a = 0, b = 0$

Orthogonal: $\vec{A} \cdot \vec{B} = 0$

Functions are similar, but not exactly the same

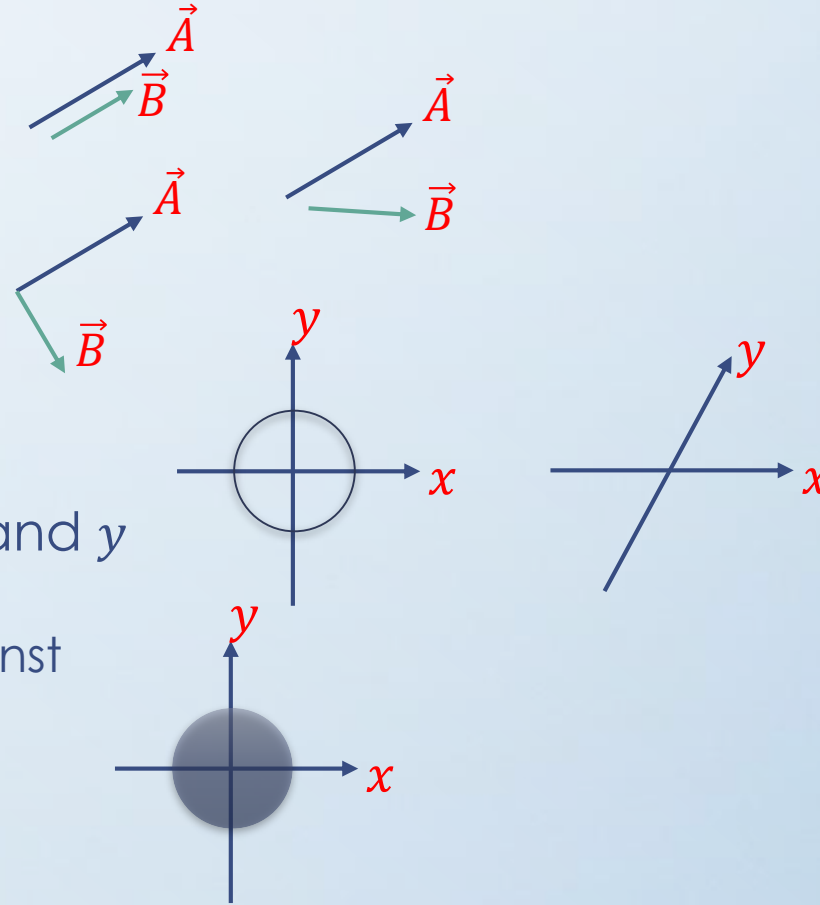
Dependent: $x, y, y = y(x)$ or $x = x(y)$

Independent: no function relation between x and y

Partial Differentiation of $f(x, y)$: $\frac{\partial f(x, y)}{\partial x}$, take y as a const

Complete Differentiation of $f(x, y)$:

$$\frac{df(x, y)}{dx} = \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y} \frac{dy}{dx}, y = y(x)$$



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9. TAYLOR EXPANSION

Any function $f(x)$ can be expanded at $x = a$ $f(x) = C_0 + C_1(x - a) + C_2(x - a)^2 + \dots$

Taylor expansion of $f(x)$ at $x = a$

$$f(x) = f(a) + \left. \frac{df(x)}{dx} \right|_{x=a} \frac{(x-a)}{1!} + \left. \frac{d^2f(x)}{dx^2} \right|_{x=a} \frac{(x-a)^2}{2!} + \dots$$

Taylor expansion of $f(x)$ at $x = 0$ $f(x) = C_0 + C_1x + C_2x^2 + \dots$

$$f(x) = f(0) + f'(0) \frac{x}{1!} + f''(0) \frac{x^2}{2!} + f'''(0) \frac{x^3}{3!} + \dots$$

For example, $f(x) = x^2 + 2x$, $f(0) = 0$, $f'(0) = 2$, $f''(0) = 2$

$$f'''(0) = f''''(0) = \dots = 0$$

Taylor expansion gives $f(x) = 0 + \frac{2}{1!}x + \frac{2}{2!}x^2$

9. TAYLOR EXPANSION

Series of e $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$

Series of e^x $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Differentiation of e^x $\frac{de^x}{dx} = 0 + \frac{1}{1!} + \frac{2x^1}{2!} + \frac{3x^2}{3!} + \dots$

$$\frac{de^x}{dx} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots = e^x$$

$\frac{d \ln x}{dx} = ?$ Let $y = \ln x \rightarrow x = e^y$ $\frac{dx}{dx} = \frac{de^y}{dx} = \frac{de^y}{dy} \frac{dy}{dx} = e^y \frac{dy}{dx}$

$$1 = e^y \frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

9. TAYLOR EXPANSION

Differentiation of $\sin(x)$:

$$\begin{aligned}\frac{d \sin(x)}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x) \cos(\Delta x) + \cos(x) \sin(\Delta x) - \sin(x)}{\Delta x} = \cos(x)\end{aligned}$$

Differentiation of $\cos(x)$:

$$\begin{aligned}\frac{d \cos(x)}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\cos(x + \Delta x) - \cos(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos(x) \cos(\Delta x) - \sin(x) \sin(\Delta x) - \cos(x)}{\Delta x} = -\sin(x)\end{aligned}$$

Taylor expansion:

$$\begin{aligned}\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\end{aligned}$$

9. TAYLOR EXPANSION

$$\sin(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots = \sin(0) + \frac{\cos(0)}{1!}x + \frac{-\sin(0)}{2!}x^2 + \frac{-\cos(0)}{3!}x^3 + \dots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \dots$$

$$\cos(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots = \cos(0) + \frac{-\sin(0)}{1!}x + \frac{-\cos(0)}{2!}x^2 + \frac{\sin(0)}{3!}x^3 + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \dots$$

Imaginary number: $i \equiv \sqrt{-1}$, or $i^2 = -1$

Notation:

$$e^{ix} = 1 + \frac{(ix)}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)$$

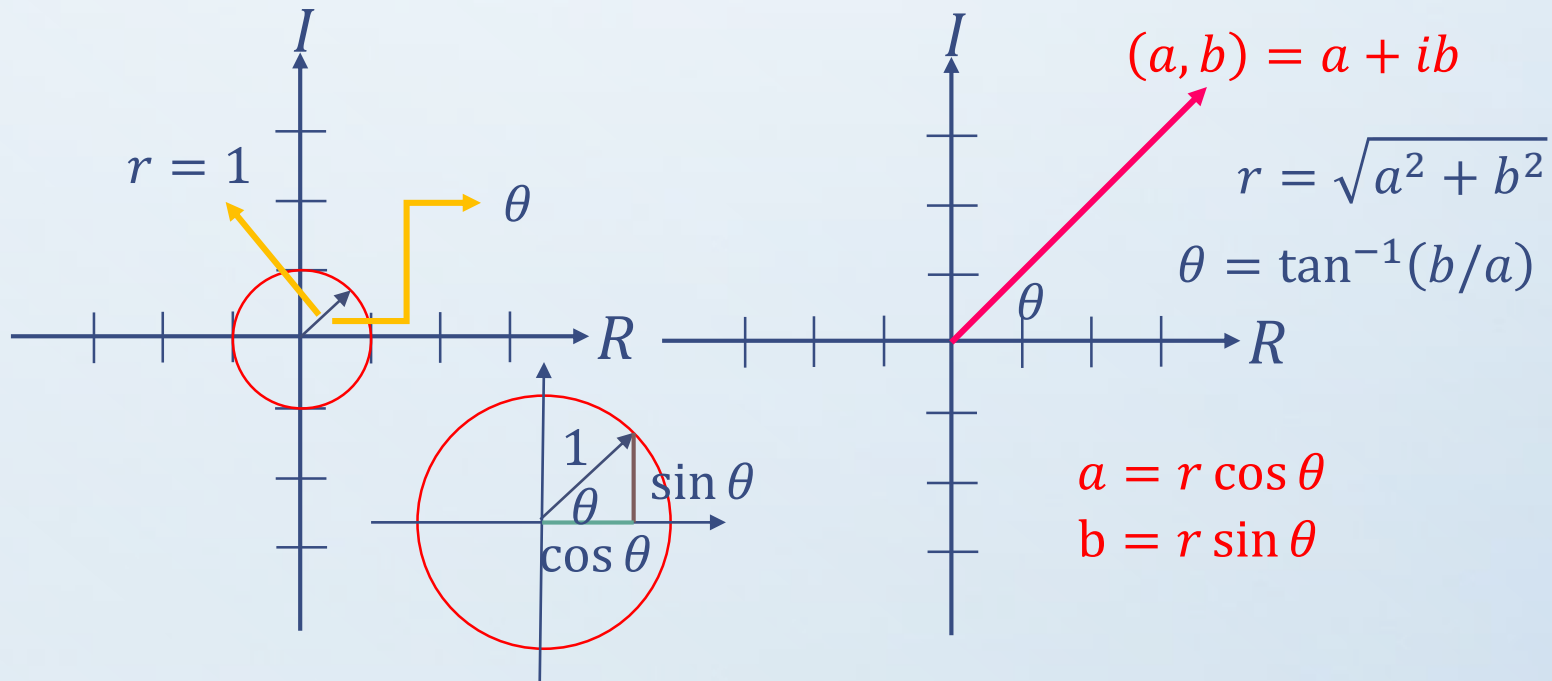
$$e^{ix} = \cos x + i \sin x$$

10. COMPLEX NUMBERS

Embodiment the notation of complex number $e^{ix} = \cos x + i \sin x$

Present in the polar coordinate $e^{i\theta} = \cos \theta + i \sin \theta$

$$re^{i\theta} = r \cos \theta + ir \sin \theta$$



10. COMPLEX NUMBERS (APPLICATION)

Trigonal functions – addition formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{i\alpha} = \cos \alpha + i \sin \alpha \quad e^{i\beta} = \cos \beta + i \sin \beta$$

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta} \quad \Rightarrow$$

$$\cos(\alpha + \beta) + i \sin(\alpha + \beta) = (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)$$

$$\cos(\alpha + \beta) + i \sin(\alpha + \beta) = (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \sin \beta \cos \alpha)$$

$$\Rightarrow \begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \sin \beta \cos \alpha \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \end{aligned}$$

Trigonal functions – double angle formula

$$e^{i(2\theta)} = e^{i\theta} e^{i\theta} \quad \Rightarrow \begin{aligned} \sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \end{aligned}$$

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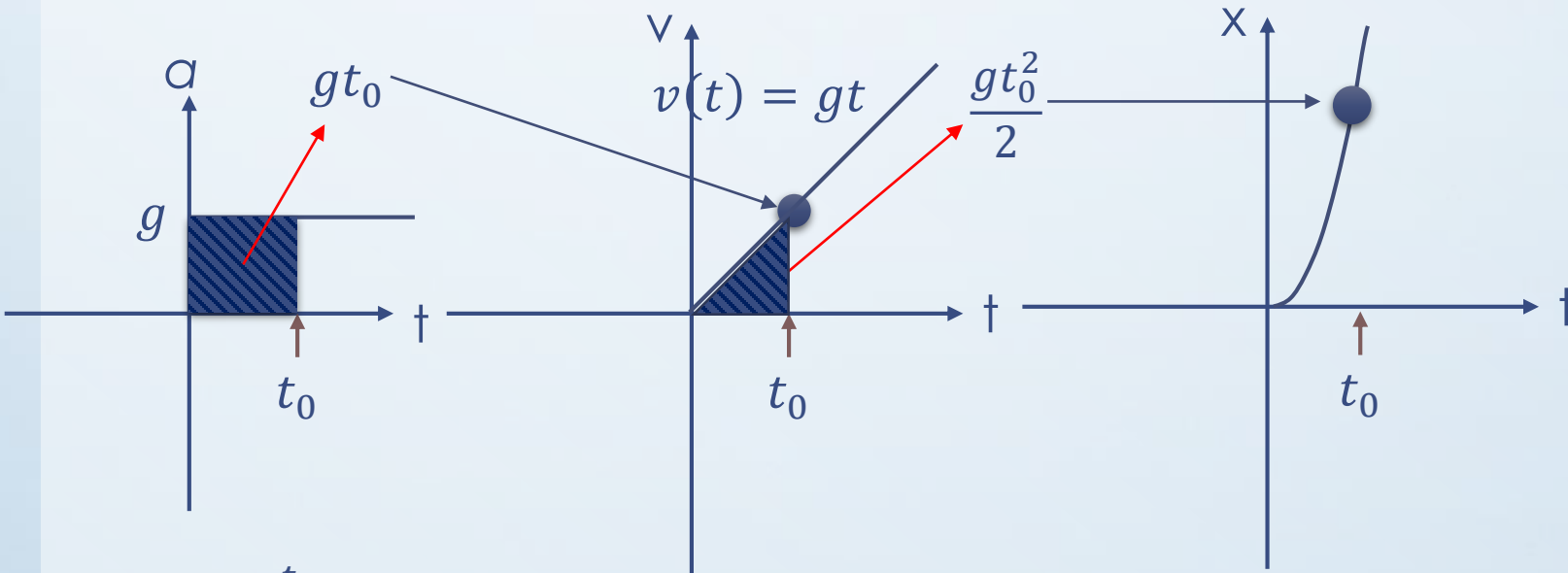
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11. INTEGRATION

constant acceleration g – **freely falling object**



$$v(t_0) = \int_0^{t_0} g dt = [gt]_{t=0}^{t=t_0} = gt_0 - g \cdot 0 = gt_0 \Rightarrow v(t) = gt$$

$$x(t_0) = \int_0^{t_0} gt dt = \left[\frac{gt^2}{2} \right]_{t=0}^{t=t_0} = \frac{gt_0^2}{2} - \frac{g \cdot 0^2}{2} = \frac{gt_0^2}{2} \Rightarrow x(t) = gt^2/2$$

11. INTEGRATION

Differentiation

$$f(x) = x^n \quad \frac{df(x)}{dx} = nx^{n-1}$$

$$f(x) = e^x \quad \frac{df(x)}{dx} = e^x$$

$$f(x) = \sin x \quad \frac{df(x)}{dx} = \cos x$$

$$f(x) = \ln x \quad \frac{df(x)}{dx} = \frac{1}{x}$$

Integration

$$f(x) = x^n, \int f(x)dx = \frac{x^{n+1}}{n+1} + c$$

$$f(x) = e^x, \int f(x)dx = e^x + c$$

$$f(x) = \cos x, \int f(x)dx = \sin x + c$$

$$f(x) = \frac{1}{x}, \int f(x)dx = \ln x + c$$

11. INTEGRATION – CHAIN RULE & CHANGE OF VARIABLES

$$\frac{d(\sin(x^2 + 1))}{dx} = \frac{d(\sin(x^2 + 1))}{d(x^2 + 1)} \frac{d(x^2 + 1)}{dx} = \cos(x^2 + 1) \times 2x$$

$$\int_0^1 2x \cos(x^2 + 1) dx = [\sin(x^2 + 1)]_{x=0}^{x=1}, \text{ how?}$$

$$\int_0^1 2x \cos(x^2 + 1) dx = \int_0^1 \cos(x^2 + 1) d(x^2) = \int_0^1 \cos(x^2 + 1) d(x^2 + 1)$$

$$= \int_0^1 d(\sin(x^2 + 1)) = [\sin(x^2 + 1)]_{x=0}^{x=1}$$

$$\int_0^{\pi/4} \frac{1}{\cos(x)} dx = \int_0^{\pi/4} \frac{\cos(x)}{\cos^2(x)} dx = \int_0^{\pi/4} \frac{1}{\cos^2(x)} d(\sin(x)) = \int_0^{\pi/4} \frac{1}{1 - \sin^2(x)} d(\sin(x))$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} \right) d(\sin(x)) = \frac{1}{2} \int_0^{\pi/4} \frac{1}{1 + \sin x} d(1 + \sin(x)) - \frac{1}{2} \int_0^{\pi/4} \frac{1}{1 - \sin x} d(1 - \sin(x))$$

$$= \frac{1}{2} [\ln((1 + \sin x)/(1 - \sin x))]_{x=0}^{x=\pi/4}$$

12. 1ST ORDER DIFFERENTIAL EQUATION

Some problems like population growth

Its growth rate is proportional to the population at that time

Assume the population P as a function of time t is expressed as $P(t)$

Its growth rate (or variation) is $dP(t)/dt$

If the growth rate is proportional to the population with a constant k

That can be expressed as $\frac{dP(t)}{dt} = kP(t)$ We obtain the 1st order DE

To solve the problem, we need to have one condition like $P(t = 0) = P_0$

Separation of variable $\frac{dP(t)}{P(t)} = k dt$

$$\int \frac{dP}{P} = \int k dt \quad \int_{P_0}^{P(t')} \frac{dP}{P} = \int_0^{t'} k dt \quad [\ln P]_{P_0}^{P(t')} = [kt]_0^{t'}$$

$$\ln(P(t')) - \ln P_0 = kt' - 0 \quad \ln\left(\frac{P(t')}{P_0}\right) = kt' \quad \frac{P(t')}{P_0} = e^{kt'} \quad P(t') = P_0 e^{kt'}$$

12. 1ST ORDER DIFFERENTIAL EQUATION

Check that $P(t) = P_0 e^{kt}$ is the solution of the 1st order DE $\frac{dP}{dt} = kP$

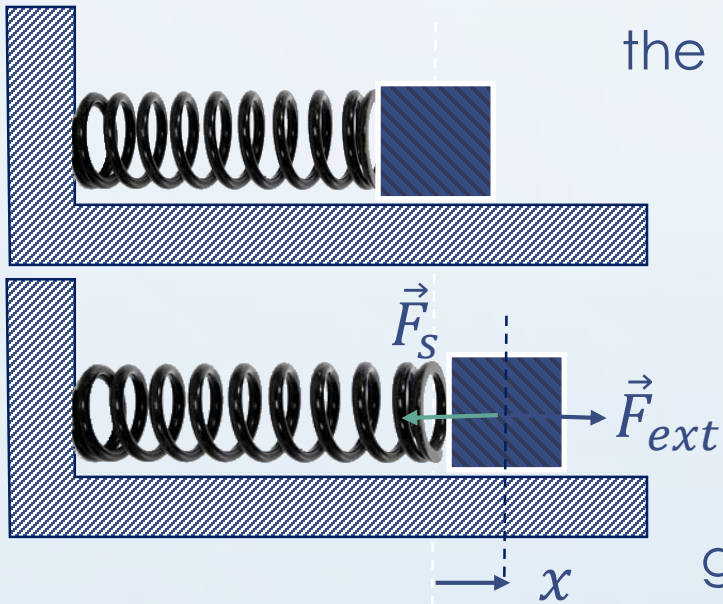
Put the solution into the DE, check if the equation is satisfied or not

$$\frac{dP(t)}{dt} = \frac{d(P_0 e^{kt})}{dt} = \frac{P_0 d(e^{kt})}{dt} = P_0 \frac{d(e^{kt})}{d(kt)} \frac{d(kt)}{dt} = kP_0 e^{kt} = kP(t)$$

The equation is satisfied by using the function $P(t)$

Thus the function $P(t) = P_0 e^{kt}$ is the solution of the DE $\frac{dP}{dt} = kP$

13. 2ND ORDER DIFFERENTIAL EQUATION



Derive the 2nd order DE from
the force equation of the spring system

$$F = -kx \quad \rightarrow \quad -kx = ma$$
$$F = ma$$

$$ma + kx = 0$$

$$\rightarrow m \frac{d^2x}{dt^2} + kx = 0$$

guess solution $x(t) = A \sin(Bt)$ or $x(t) = A \cos(Bt)$

$$-mB^2A \sin(Bt) + kA \sin(Bt) = 0$$

$$(k - mB^2)A \sin(Bt) = 0$$
$$B = \sqrt{\frac{k}{m}} \quad x(t) = A \sin(\sqrt{k/m}t)$$

13. 2ND ORDER DIFFERENTIAL EQUATION

$$m \frac{d^2 x}{dt^2} + kx = 0$$

guess solution $x(t) = Ae^{Bt}$, put the guess solution into the DE

$$mB^2 Ae^{Bt} + kAe^{Bt} = 0$$

$$(k + mB^2)Ae^{Bt} = 0 \quad \longrightarrow \quad B = i\sqrt{k/m} \quad B = -i\sqrt{k/m}$$

$$x(t) = A_1 e^{i\sqrt{k/m}t} + A_2 e^{-i\sqrt{k/m}t} = C_1 \sin(\sqrt{k/m}t) + C_2 \cos(\sqrt{k/m}t)$$

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