Physics I
Lecture07- Energy of a System-I

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CONTENTS

USING ENERGY TO SOLVE PROBLEM

Energy concept used for solving physical problems is essential, especially for the cases of conservative forces.

For example, the spring system, the force equation is

$$\vec{F} = -k\vec{x} = m\vec{a} = m\frac{d^2\vec{x}}{dt^2}$$

You may need to solve the differential equation if you work on force. On the other hand, the force of the spring system is conservative so it can be expressed by the new concept of potential energy.

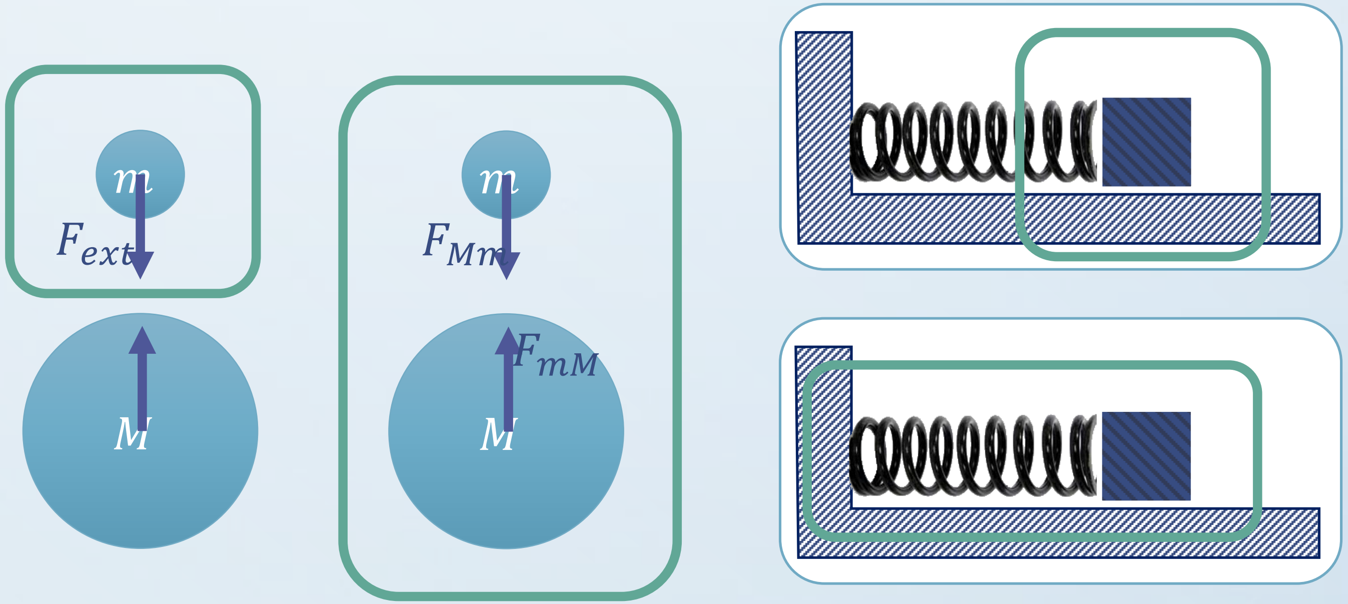
$$U = -\int \vec{F} \cdot d\vec{r} = -\int -kxdx = \frac{kx^2}{2} + c$$

Use the new concept of energy conservation, you can easily solve some physical problems.

$$E = \frac{kx^2}{2} + \frac{mv^2}{2}$$

1. SYSTEMS & ENVIRONMENTS

Understand the system with its environment. A system may
1. be a single objet (particle)
2. be a collection of objects (particles)
3. be a region of space (filling with gas or liquid, such as an engine)
4. vary in size and shape (rubber ball)



1. SYSTEMS & ENVIRONMENTS

Example: A block with mass m_1 is sitting on frictionless surface on the top of a car of mass M. It is connected to another block of mass m_2 by a lightweight cord over a lightweight and frictionless pulley. Assume that the car can move without friction. At the instant the two blocks start to move, what are the accelerations of the two blocks and the car?

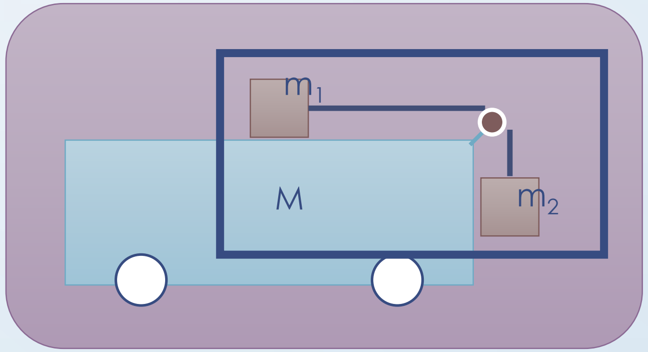
$$m_2 g - T = m_2 a$$

$$T = m_1(a - A)$$

$$T = MA$$

$$a = \frac{(M + m_1)m_2 g}{Mm_1 + m_1 m_2 + m_2 M}$$

$$A = \frac{m_1 m_2 g}{Mm_1 + m_1 m_2 + m_2 M}$$



2. THE SCALAR PRODUCT OF VECTORS

Start from the x, y, z unit vectors, the product is a projection length of a vector on another vector: $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1,\ \hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$

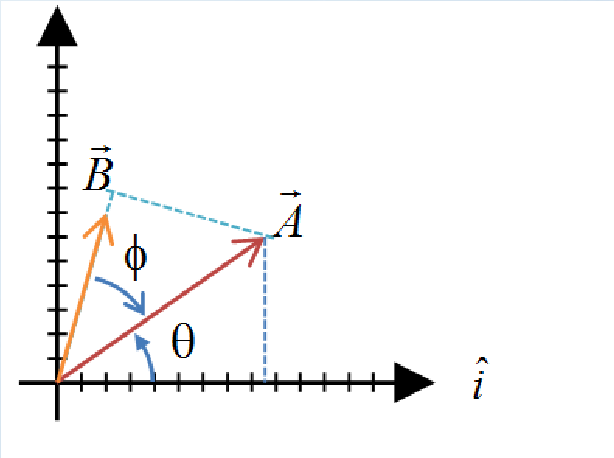
Calculation $\vec{A}\cdot\vec{B}$ from the components: $\vec{A}\cdot\vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)\cdot\left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right) = A_xB_x + A_yB_y + A_zB_z$

$$\cos(\phi+\theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$
$$\sin(\phi+\theta) = \sin\phi\cos\theta + cos\,\phi\sin\theta$$

$$\frac{B_x}{B} = \cos\phi\frac{A_x}{A} - \sin\phi\frac{A_y}{A}$$
$$\frac{B_y}{B} = \sin\phi\frac{A_x}{A} + \cos\phi\frac{A_y}{A}$$

$$\frac{A_xB_x}{B} + \frac{A_yB_y}{B} = \cos\phi\, A$$

$$\cos\phi = \frac{A_xB_x + A_yB_y}{AB} \Rightarrow \vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos(\phi) = AB\cos\phi$$



2. THE SCALAR PRODUCT OF VECTORS

Example: The vectors $\vec{A}$ and $\vec{B}$ are given as $\vec{A} = 3\hat{\imath} + 2\hat{\jmath}$, $\vec{B} = -\hat{\imath} + 2\hat{\jmath}$. (a) Determine their scalar product. (b) Find the angle between the two vectors.

(a)

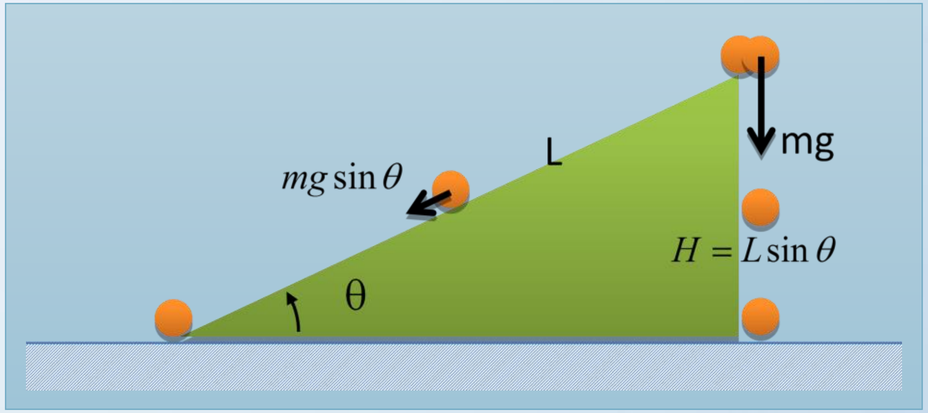
$$\vec{A} \cdot \vec{B} = 3 \times (-1) + 2 \times 2 = 1$$

(b)

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{1}{\sqrt{3^2 + 2^2}\sqrt{(-1)^2 + 2^2}} = \frac{1}{\sqrt{65}}$$

$$\theta \cong 82.9^0$$

3. WORK DONE BY A CONSTANT FORCE



Work: $W = FS$
Vertical direction: $W = mgH$
On the slide: $W = mg\sin\theta\, L$
$= mgL\cos\left(\frac{\pi}{2} - \theta\right) = \vec{F}\cdot\vec{L}$

Summary, Work: $W = \vec{F}\cdot\vec{S}$, The Unit of Work: 1 N × 1 m = 1 J (Joule) done on the particle.

Example: A partile moving on the xy plane undergoes a displacement $\Delta\vec{r} = 2.0\hat{\imath} + 3.0\hat{\jmath}$ (m) when it is exerted by a constant force $\vec{F} = 5.0\hat{\imath} + 2.0\hat{\jmath}$. Please calculate the work

$W = \vec{F}\cdot\Delta\vec{r} = (2.0\times 5.0) + (3.0\times 2.0) = 16$ (J)

3. WORK DONE BY A CONSTANT FORCE

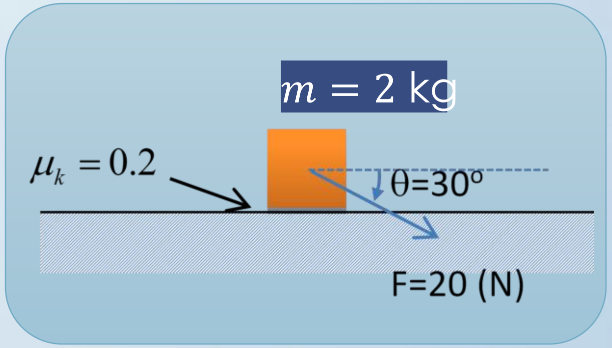
Example: A block of mass 2.00 kg on a table is pushed by a force of 20.0 N directed with an angle of 30º downward from the horizontal. The kinetic frictional coefficient is 0.200. Please calculate the work done by the force and friction after traveling 10.0 m long.

$N = mg + F\sin 30^0 = 19.6 + 10.0 = 29.6$ (N)

$f_k = N\mu_k = 29.6 \times 0.2 = 5.92$ (N)

Work Done by The Force: $FS\cos 30^0 = 20.0 \times 10.0 \times \frac{\sqrt{3}}{2} \cong 173$ (J)

Work Done by Friction: $f_k S = 5.92 \times 10.0 = 59.2$ (J)



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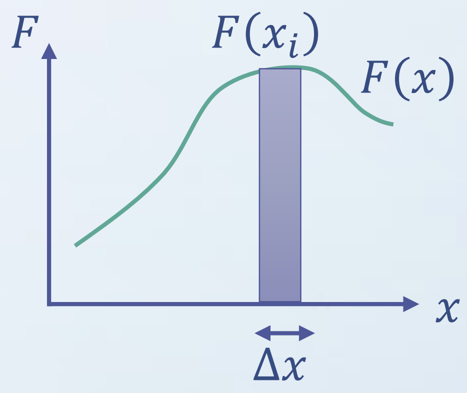
4. WORK DONE BY A VARYING FORCE

Varying Force: $F(x_i) \neq F(x_j)$



$$\Delta W_i = F(x_i)\Delta x \Rightarrow W = \sum_{i=1}^{N} \Delta W_i = \sum_{i=1}^{N} F(x_i)\Delta x$$

$$\lim_{N\to\infty} \sum_{i=1}^{N} F\left(x_i + i\,\frac{x_f - x_i}{N}\right)\frac{x_f - x_i}{N} = \int_{x_i}^{x_f} F(x)dx$$

$$W = \int dW \qquad dW = Fdx = \vec{F}(\vec{r}) \cdot d\vec{r}$$

$$W = \int_{\vec{r}_i}^{\vec{r}_j} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{x_i,y_i,z_i}^{x_f,y_f,z_f} F_x(x,y,z)dx + \int_{x_i,y_i,z_i}^{x_f,y_f,z_f} F_y(x,y,z)dy$$
$$+ \int_{x_i,y_i,z_i}^{x_f,y_f,z_f} F_z(x,y,z)dz$$

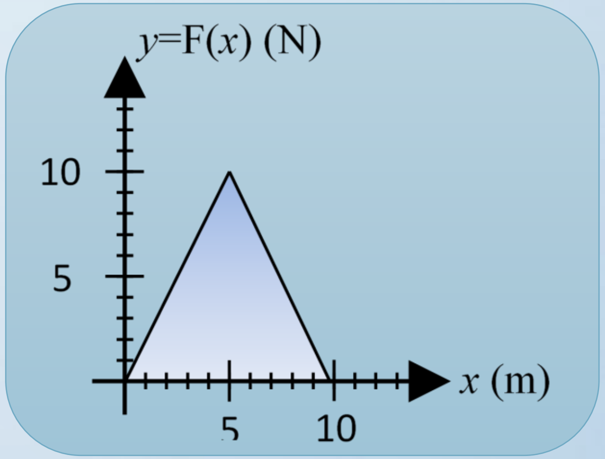
4. WORK DONE BY A VARYING FORCE

Example: In the graph, the force acting on an particle is shown as a function of position. If the particle is exerted by the force and moving from $x = 0$ to $x = 10$ m, please calculate the work done on it.

$$W = \int F dx$$

It is noted that the integration is just the area on the $F - x$ graph.

$$W = Area = \frac{1}{2} 10 \times 10 = 50 \text{ (J)}$$



4. WORK DONE BY A VARYING FORCE

Example: A force of $\vec{F} = 3x^2\hat{\imath} + 4\hat{\jmath}$ N, with x in meters, acts on a particle, changing only the kinetic energy of the particle. How much work is done on the particle as it moves from (2, 3) to (3, 0) (m)? Does the speed of the particle increase, decrease, or remain the same?

$$W = \int_{2\hat{\imath}+3\hat{\jmath}}^{3\hat{\imath}} \left(3x^2\hat{\imath} + 4\hat{\jmath}\right) \cdot \left(\hat{\imath}dx + \hat{\jmath}dy\right)$$

$$W = \int_{2\hat{\imath}+3\hat{\jmath}}^{3\hat{\imath}} \left(3x^2dx + 4dy\right) = \int_{2}^{3} 3x^2dx + \int_{3}^{0} 4dy$$

$$W = [x^3]_{x=2}^{x=3} + [4y]_{y=3}^{y=0} = 19 - 12 = 7 \text{ (J)}$$

The work is positive thus the energy of the particle increases. The speed of the particle increases.

4. WORK DONE BY A VARYING FORCE

Work Done on The Block by a Spring

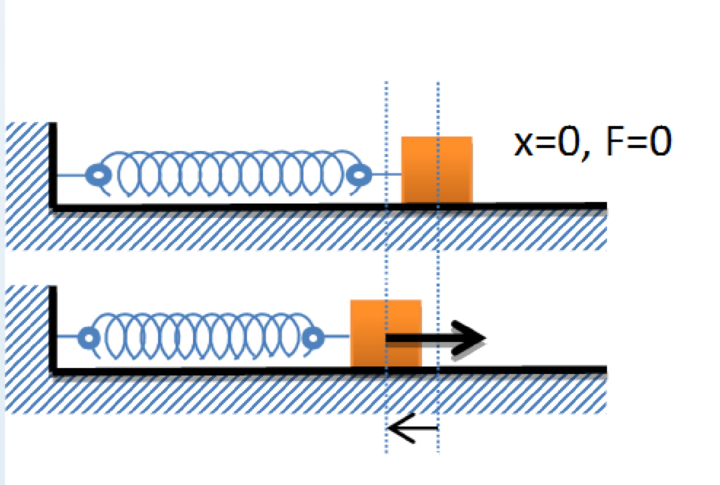
$$W = \int_{x_i}^{x_f} (-kx)dx = -\left[\frac{kx^2}{2}\right]_{x_i}^{x_f}$$

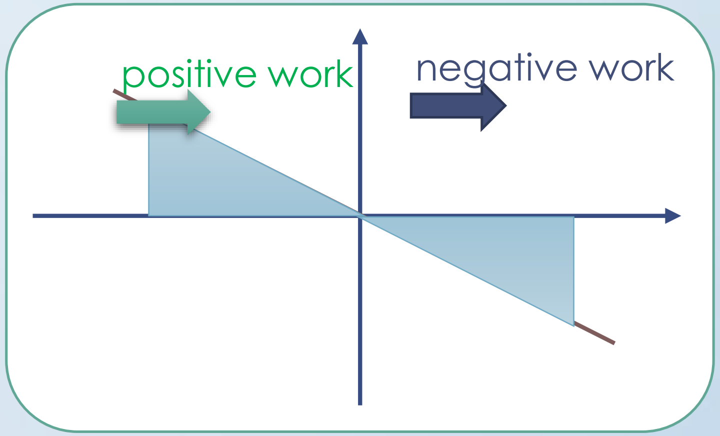
$$W = \frac{kx_i^2}{2} - \frac{kx_f^2}{2}$$

$$W = \int_0^x (-kx)dx = -\frac{kx^2}{2}$$

$$W = \int_0^{x_{max}} (-kx)dx = -\frac{kx_{max}^2}{2}$$

$$W = \int_{-x_{max}}^0 (-kx)dx = \frac{kx_{max}^2}{2}$$





4. WORK DONE BY A VARYING FORCE

Example: An archer pulls her bowstring back 0.400 m by exerting a force that increases uniformly from zero to 230 N.
(a) What is the equivalent spring constant of the bow? (b) How much work does the archer do in pulling the bow?

The force increases uniformly, thus it follow the expression $F(x) = -kx$.

$230 = k \times 0.400 \Rightarrow k = 575$ (N/m)

The archer did positive work, $W = \frac{1}{2}kx^2$

$W = \frac{575}{2}(0.400)^2 = 46.0$ (J)

4. WORK DONE BY A VARYING FORCE

Example: A small particle of mass m is pulled to the top of a frictionless half-cylinder (of radius R) by a cord that passes over the top of the cylinder. (a) If the particle moves at a constant speed, calculate the tension in the cord. (b) Find the work done in moving the particle at constant speed from the bottom to the top of the half-cylinder.

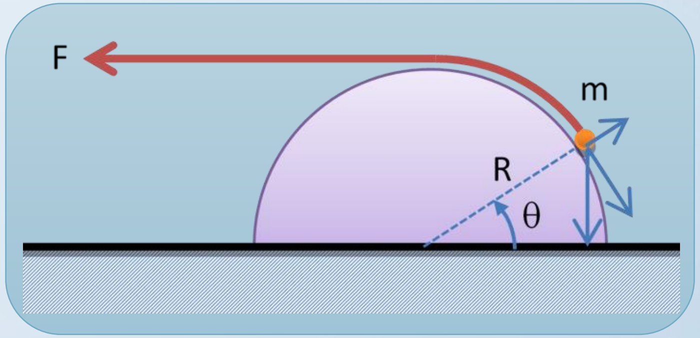
(a)

$$F = T = mg\cos\theta$$

(b)

$$W = \int F_x dx = \int F_s ds = \int_0^{\pi/2} (mg\cos\theta)(Rd\theta)$$

$$W = mgR[\sin\theta]_{\theta=0}^{\theta=\pi/2} = mgR$$



5. WORK-KINETIC ENERGY THEOREM

The kinetic energy is an inertia energy of the system. For a motional particle, the kinetic energy is just a function of its mass and speed.

$$K = \frac{mv^2}{2}$$

Work is a mechanism to transfer energy into or out of a system. You can do positive work to raise the system energy, or you can do negative work to decrease the system energy.

$$W = \int F(x)dx \qquad F = ma = m\frac{dv}{dt} \qquad W = \int m\frac{dv}{dt}dx$$

$$dx = vdt \qquad W = \int m\frac{dv}{dt}vdt \qquad W = \int \frac{d(mv^2/2)}{dt}dt$$

$$W = \int d\left(\frac{mv^2}{2}\right) = \int_{v_i}^{v_f} d\left(\frac{mv^2}{2}\right) = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} = \Delta K$$

5. WORK-KINETIC ENERGY THEOREM

Example: A 6.0 kg block initially at rest is pulled to the right along a horizontal, frictionless surface by a constant force of 12 N. Find the speed of the block after it has moved 3.0 m.

Use The Force Concept to Solve It:

$$a = \frac{12}{6.0} = 2.0\ (\mathrm{m/s^2}) \qquad v^2 = v_0^2 + 2as = 0 + 2 \times 2.0 \times 3.0 = 12$$

$$v = 3.5\ (\mathrm{m/s})$$

Use The Energy Concept to Solve It:

$$W = FS = 12 \times 3.0 = 36\ (\mathrm{J})$$

$$W = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} \Rightarrow 36 = \frac{6.0 \times v_f^2}{2} \qquad v_f = 3.5\ (\mathrm{m/s})$$

6. POTENTIAL ENERGY

Potential Energy – convert the external force to energy and put the converted energy to the system energy

$$W = \int_i^f \vec{F}_{ext} \cdot d\vec{r} = W_f - W_i$$

$$W_f - W_i = \Delta K = K_f - K_i \Rightarrow K_f - W_f = K_i - W_i$$

$$W = -U \Rightarrow -U_f + U_i = K_f - K_i \Rightarrow E = K_i + U_i = K_f + U_f$$

Potential Energy of Gravitational Force

$$\vec{F} = -mg\hat{j} \qquad U = -W = -\int_{y_0}^{y_0+h} (-mg\hat{j}) \cdot (\hat{j}dy)$$

$$U = \int_{y_0}^{y_0+h} mgdy = mgh$$

6. POTENTIAL ENERGY

Potential Energy of Gravitational Force, Considering The High Altitude Effect:

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

The Displacement in The 3D Space by The Spherical Coordinate:

$$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz = \hat{r}dr + \hat{\theta}(rd\theta) + \hat{\phi}(r\sin(\theta))d\phi$$

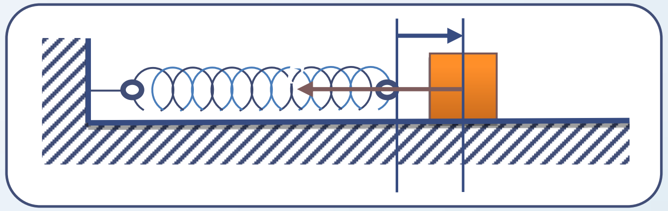
$$U = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = -\int_{\vec{r}_1}^{\vec{r}_2} \left(-\frac{GMm}{r^2}\hat{r}\right)\cdot\left(\hat{r}dr + \hat{\theta}(rd\theta) + \hat{\phi}(r\sin(\theta))d\phi\right)$$

$$U = GMm\int_{r_1}^{r_2}\frac{dr}{r^2} = GMm\left[-\frac{1}{r}\right]_{r_1}^{r_2} = -\frac{GMm}{r_2} + \frac{GMm}{r_1}$$

Choose $r_1 \to \infty$ $\quad U = -\dfrac{GMm}{r_2}$

6. POTENTIAL ENERGY

Potential Energy of a Spring



$$\vec{F} = -k\vec{x} = -kx\hat{\imath}$$

$$U = -W = -\int_{x_i}^{x_f} (-kx\hat{\imath}) \cdot (\hat{\imath}dx) = \frac{kx_f^2}{2} - \frac{kx_i^2}{2}$$

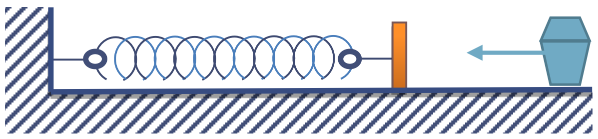
$$W = \Delta K = K_f - K_i \qquad U = \frac{kx_f^2}{2} - \frac{kx_i^2}{2} = -K_f + K_i$$

$$E = K_i + U_i = K_f + U_f = \frac{mv_i^2}{2} + \frac{kx_i^2}{2} = \frac{mv_f^2}{2} + \frac{kx_f^2}{2}$$

$$= \frac{mv^2}{2} + \frac{kx^2}{2}$$

6. POTENTIAL ENERGY

Example: A canister of mass m = 0.40 kg slides across a horizontal frictionless counter with speed v = 0.50 m/s. It then runs into and compresses a spring of spring constant k = 750 N/m. When the canister is momentarily stopped by the spring, by what distance d is the spring compressed?



The kinetic energy: $\frac{mv^2}{2} = \frac{0.40 \times (0.50^2)}{2} = 0.050$ (J)

The energy is used to compress the spring and change to potential energy.

$$\frac{kd^2}{2} = \frac{750 \times d^2}{2} = 0.050 \Rightarrow d = 0.012\ (\mathrm{m}) = 1.2\ (\mathrm{cm})$$

CONTENTS

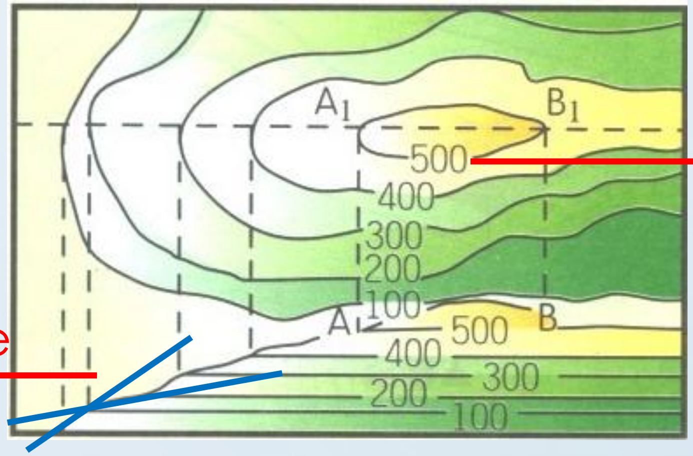
7. CONSERVATIVE FORCES

Conservative Forces

1. The work done by a conservative force on a particle between two different position is independent of the path taken by the particle.

2. The work done by a conservative force on a particle moving through any closed loop is zero.

3. There exists a scalar potential that its gradient is the conservative force. -> vector calculus – gradient theory



slope -> force

scarlar potential

force – slope:

$$\frac{\Delta \mathrm{U}}{\Delta x} = \frac{\Delta h}{\Delta x}$$

7. CONSERVATIVE FORCES

Example: A single constant force $\vec{F} = 3\hat{\imath} + 5\hat{\jmath}$ N acts on a 4.00-kg particle. (a) Calculate the work done by this force if the particle moves from the origin O to a position P at $3\hat{\imath} - 2\hat{\jmath}$ m. Does this result depend on the path? (b) What is the speed of the particle at P if its speed at O is 4.00 m/s? (c) What is the change in its potential energy?

(a)

$$W_1 = \int_{0,0}^{3,-2} (3\hat{\imath} + 5\hat{\jmath}) \cdot (\hat{\imath}dx + \hat{\jmath}dy) = \int_{0,0}^{3,-2} (3dx + 5dy) = -1 \text{ (J)}$$

$$W_2 = \int_{0,0}^{3,0} (3dx + 5dy) + \int_{3,0}^{3,-2} (3dx + 5dy) = -1 \text{ (J)}$$

(b)

$$W = \Delta E \qquad -1 = \frac{4.00 \times v^2}{2} - \frac{4.00 \times 4.00^2}{2} \qquad v = 3.94 \text{ (m/s)}$$

(c)

$$U = -W = 1 \text{ (J)}$$

8. RELATION BETWEEN POTENTIAL ENERGY AND CONSERVATIVE FORCES

The relation between $\vec{F}$ and U $\quad U = -\int F dx = -\int \vec{F} \cdot d\vec{r}$

$$U = \int dU \quad \Rightarrow \quad dU = -F dx \quad \Rightarrow \quad F = -\frac{dU(x)}{dx}$$

$$dU = \vec{F} \cdot d\vec{r} = -\left(F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k}\right) \cdot \left(\hat{\imath} dx + \hat{\jmath} dy + \hat{k} dz\right)$$

$$dU(x, y, z) = -F_x dx - F_y dy - F_z dz$$

Since x, y, z are independent $\quad F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$

$$\vec{F}(x, y, z) = F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k} = -\frac{\partial U}{\partial x} \hat{\imath} - \frac{\partial U}{\partial y} \hat{\jmath} - \frac{\partial U}{\partial z} \hat{k}$$

$$\vec{F}(x, y, z) = -\left(\hat{\imath} \frac{\partial}{\partial x} + \hat{\jmath} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) U(x, y, z) = -\vec{\nabla} U(x, y, z)$$

8. RELATION BETWEEN POTENTIAL ENERGY AND CONSERVATIVE FORCES

The Potential Energy and Force of a Spring Sytem

$$U = \frac{1}{2}kx^2 \qquad F = -\frac{dU}{dx} = -kx$$

The Gravitational Potential Energy and Force

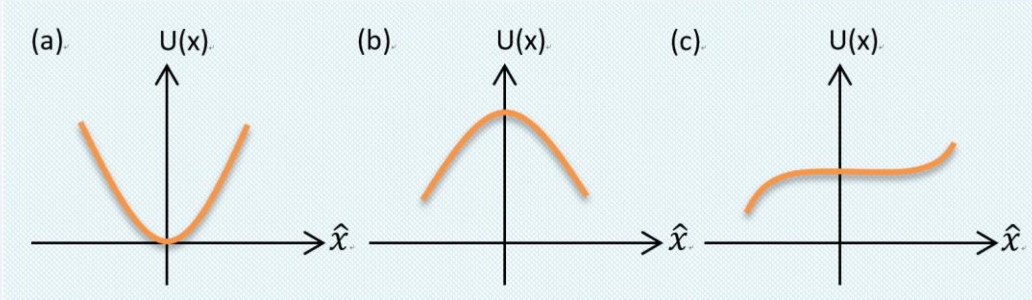
$$U = -\frac{GMm}{r} \qquad F_r = -\frac{\partial U}{\partial r} = -\frac{GMm}{r^2} \qquad \vec{F} = -\frac{GMm}{r^2}\hat{r}$$

$$U = -\int_{r=\infty}^{r'} \vec{F} \cdot d\vec{r} = -\int_{r=\infty}^{r'} \left(-\frac{GMm}{r^2}\hat{r}\right) \cdot \left(\hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta\, d\phi\right)$$

$$U = \int_{r=\infty}^{r'} \frac{GMm}{r^2}dr = \left[-\frac{GMm}{r}\right]_{r=\infty}^{r=r'} = -\frac{GMm}{r'}$$

9. ENERGY DIAGRAM AND EQUILIBRIUM

A particle is in equilibrium if the net force acting on it is zero.



(a) Stable Equilibrium: a small displacement in any direction results in a restoring force that pushes the particle back to its equilibrium position.

(b) Unstable Equilibrium: a small displacement results in a force that accelerates the particle away from its equilibrium position.

(c) Neutral Equilibrium: a small displacement results in zero force and the particle remains in equilibrium.

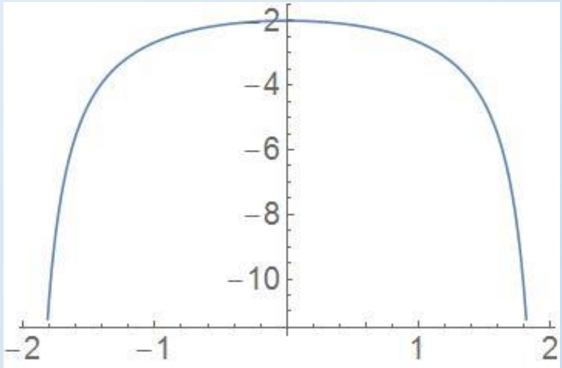
9. ENERGY DIAGRAM AND EQUILIBRIUM

Example: In the region –a < x < a the force on a particle is represented by the potential energy function $U(x) = -b\left(1/(a+x) + 1/(a-x)\right)$, where a and b are positive constants. (a) Find the force. (b) At what value of x is the force zero? (c) Is the potential stable or not?

$$F(x) = \frac{4abx}{(a^2 - x^2)^2}$$

$x = 0 \rightarrow F = 0$

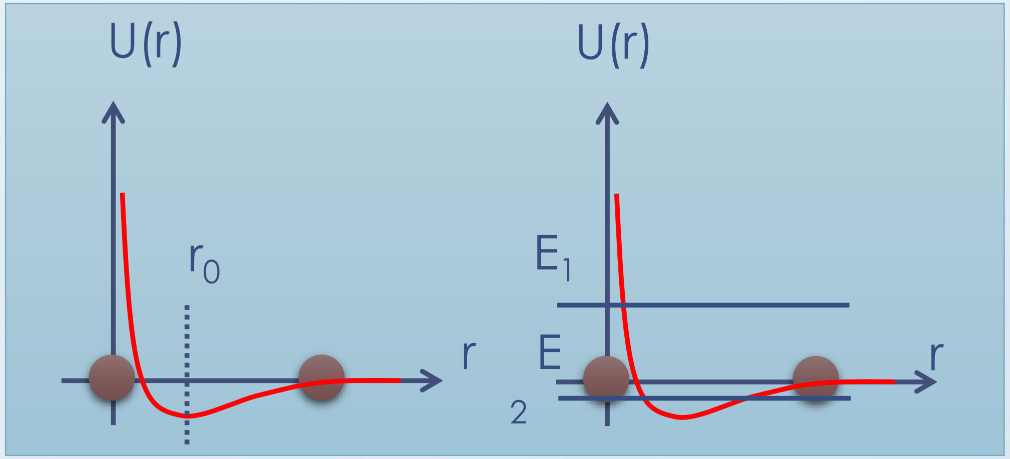
$$\left.\frac{d^2U}{dx^2}\right|_{x=0} = -\left.\frac{4ab\left(a^2 + 3x^2\right)}{(a^2 - x^2)^3}\right|_{x=0} < 0$$



9. ENERGY DIAGRAM AND EQUILIBRIUM

Example: For a Lenard-Jones potential energy of $U(x) = 4\varepsilon[(\sigma/x)^{12} - (\sigma/x)^{6}]$, where σ and ε are two parameters, please calculate its force.

$$F = -\frac{dU}{dx} = -4\varepsilon\left[-\frac{12}{x}\left(\frac{\sigma}{x}\right)^{12} + \frac{6}{x}\left(\frac{\sigma}{x}\right)^{6}\right]$$



r_0: equilibrium position
E_1: free
E_2: bounded

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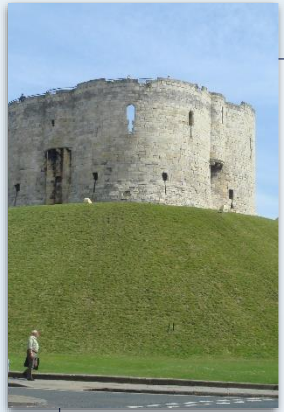

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