Chapter 11 Conservation of Angular Momentum-I

簡紋濱
國立交通大學 理學院 電子物理系

CONTENTS

1. VECTOR PRODUCT

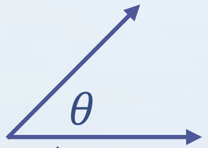
Vector Product, Defined by The Three Unit Vectors

$$\hat\imath \times \hat\imath = \hat\jmath \times \hat\jmath = \hat k \times \hat k = 0 \qquad \hat\imath \times \hat\jmath = \hat k \qquad \hat\jmath \times \hat k = \hat\imath \qquad \hat k \times \hat\imath = \hat\jmath$$

$$\vec A = A_x\hat\imath + A_y\hat\jmath + A_z\hat k,\ \vec B = B_x\hat\imath + B_y\hat\jmath + B_z\hat k$$

$$\vec A \times \vec B = \left(A_xB_y - A_yB_x\right)\hat k + \left(A_yB_z - A_zB_y\right)\hat\imath + (A_zB_x - A_xB_z)\hat\jmath \quad \Rightarrow \quad \vec A \times \vec B = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec B = B\cos\theta\,\hat\imath + B\sin\theta\,\hat\jmath$$



$\vec A = A\hat\imath$

$$\vec A \times \vec B = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ A & 0 & 0 \\ B\cos\theta & B\sin\theta & 0 \end{vmatrix} = AB\sin\theta\,\hat k$$

Some Rules: $\vec A \times \vec A = 0,\ \vec A \times \vec B = -\vec B \times \vec A$

$$\vec A \times \left(\vec B + \vec C\right) = \vec A \times \vec B + \vec A \times \vec C \qquad \frac{d}{dt}\left(\vec A \times \vec B\right) = \frac{d\vec A}{dt} \times \vec B + \vec A \times \frac{d\vec B}{dt}$$

2. ANGULAR MOMENTUM & TORQUE

Angular Momentum & Torque About an Axis

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \vec{v} \times \vec{p} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$$

$$\vec{L}_i = \vec{r}_i \times (m_i \vec{v}_i) \quad \vec{r}_i \perp \vec{v}_i$$

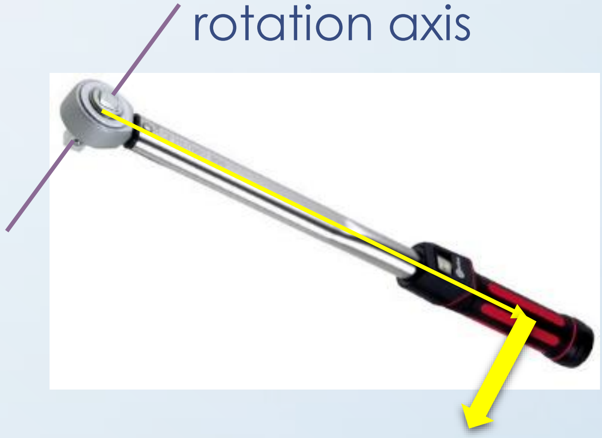
$$L_i = m_i r_i v_i \qquad v_i = r_i \omega$$

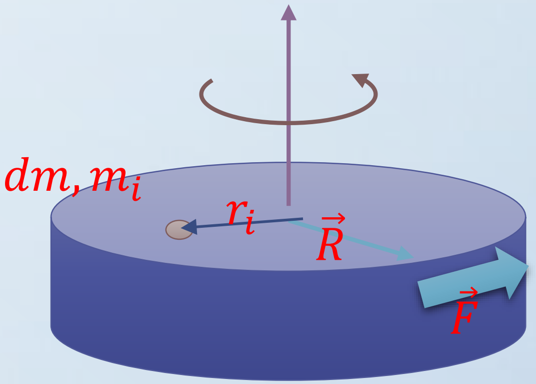
$$L = \sum L_i = \sum m_i r_i^2 \omega = I\omega$$

$$\vec{R} \times \vec{F} = \vec{\tau}_{net} = \sum \vec{\tau}_i = \frac{d}{dt}\vec{L}$$

If the direction of the angular momentum does not change, the angular acceleration is

$$\tau_{net} = \frac{d}{dt}L = \frac{d}{dt}(I\omega) = I\alpha$$





2. ANGULAR MOMENTUM & TORQUE

Example: Estimate the magnitude of the angular momentum of a basketball spinning at 8 rev/s. A typical basketball might have a mass of 6.40 kg and a radius of 10.8 cm.

$$\omega = 8\frac{rev}{s} = 16\pi\frac{rad}{s}$$

$$I_{ball} = \frac{2}{5}MR^2 = \frac{2}{5}(6.4)(0.108)^2 = 0.0299(kg \cdot m^2)$$

$$L = I_{ball}\omega = 0.0299 \times 16\pi = 1.50(kg \cdot m^2/s)$$

https://giphy.com/gifs/harlemglobetrotters-harlem-globetrotters-3o7btSjnYGCtxVIGkM

2. ANGULAR MOMENTUM & TORQUE

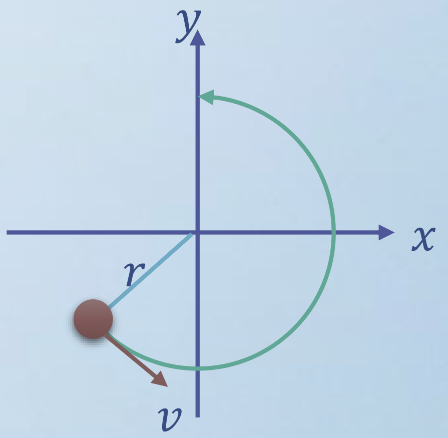
Angular Momentum & Torque of a System of Particles

$$\vec{\tau}_{net} = \frac{d}{dt}\vec{L}_{system} = \frac{d}{dt}\left(\sum \vec{r}_i \times \vec{p}_i\right)$$

Example: A particle moves on the xy plane in a circular path of radius r. Find the magnitude and direction of its angular momentum relative to O when its linear velocity is v.

$$L = rmv$$

$$\vec{L} = \vec{r} \times \left(mv\hat{\theta}\right) = rmv\hat{k}$$



2. ANGULAR MOMENTUM & TORQUE

Example: Use torque and angular momentum to calculate the acceleration of the system shown in the figure. The rotation axis is the axis of the pulley. The pulley is a disc with a radius of R and mass of M.

$$m_1 g - T_1 = m_1 a$$

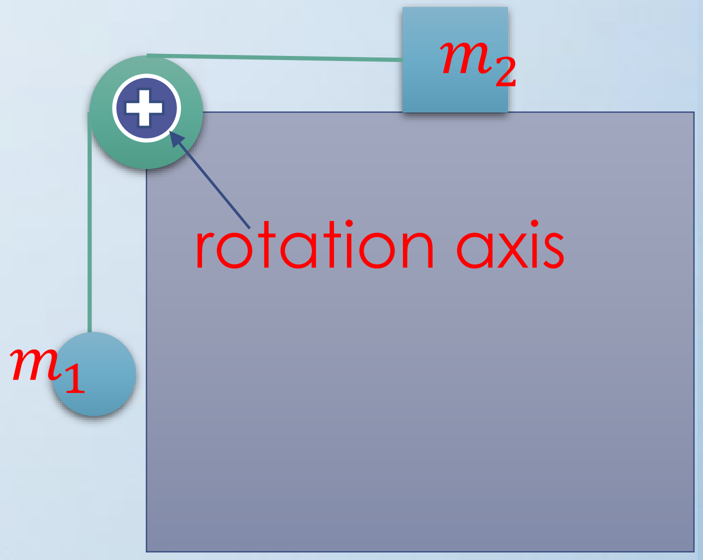
$$R(T_1 - T_2) = I\alpha, I = MR^2/2 \quad T_2 = m_2 a$$

$$a = \frac{m_1 g}{m_1 + m_2 + I/R^2} = \frac{m_1 g}{m_1 + m_2 + M/2}$$

$$L = I\omega + Rm_2 v + Rm_1 v,\ v = R\omega$$

$$\tau = Rm_1 g = \frac{d}{dt}L = \frac{d}{dt}\left(\frac{I}{R}v + Rm_1 v + Rm_2 v\right) = \left(\frac{MR}{2} + m_1 R + m_2 R\right)a$$

$$a = \frac{m_1 g}{m_1 + m_2 + M/2}$$



CONTENTS

3. CONSERVATION OF ANGULAR MOMENTUM

Total Angular Momentum

$$\vec{L}_{system} = \vec{L}_{orbital} + \vec{L}_{self\ rotation(spin)}$$

If $\vec{\tau}_{net} = 0$, the angular momentum is conserved. $\Delta\vec{L} = 0$

$$\vec{L} = L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x = m(xv_y - yv_x)$$

3. CONSERVATION OF ANGULAR MOMENTUM

Example: A man is sitting on a stool that can rotate freely about a vertical axis. The man, initially at rest, is holding a bicycle wheel whose rotational inertia about its central axis is 1.2 kg·m^2. The wheel is rotating at an angular speed of 3.9 rev/s; as seen from overhead, the rotation is counterclockwise. The man with rotational inertia 6.8 kg·m^2 now inverts the wheel so that, as seen from overhead, it is rotating clockwise. What's the angular speed of the man?

$$L_{wheel,i} + L_{m,i} = L_{wheel,f} + L_{m,f}$$

$$I_{wheel}\omega_{wheel} + 0 = -I_{wheel}\omega_{wheel} + I_m\omega_m$$

$$1.2 \times (3.9 \times 2\pi) = -1.2 \times (3.9 \times 2\pi) + 6.8 \times \omega_m$$

$$\omega_m = 1.38\left(\frac{rev}{s}\right) = 8.67\left(\frac{rad}{s}\right)$$

https://giphy.com/gifs/conservation-eaCZUjCN91q92

3. CONSERVATION OF ANGULAR MOMENTUM

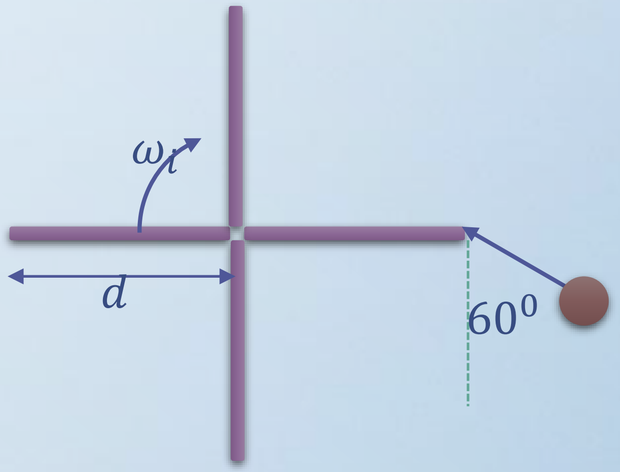
Example: Four thin, uniform rods, each of mass M and length d = 0.5 m, are rigidly connected to a vertical axle to form a turnstile. The turnstile rotates clockwise about the axle, which is attached to a floor, with initial angular velocity ω_i = -2.00 rad/s. A mud ball of mass m = M/3 and initial speed v_i = 12 m/s is thrown along the path shown and sticks to the end of one rod. What is the final angular velocity of the ball–turnstile system?

$$L_{t,i} + L_{b,i} = L_{t+b,f}$$

$$I_t = 4\left[\frac{M}{12}d^2 + M\left(\frac{d}{2}\right)^2\right] = \frac{4Md^2}{3}$$

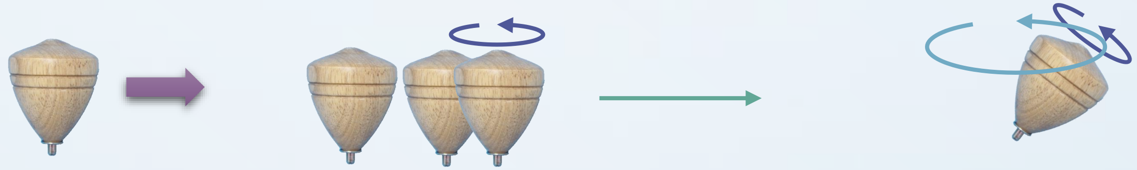
$$\frac{4}{3}Md^2(-2.00) + \frac{d}{2}\frac{M}{3}(12) = \left(\frac{4}{3}Md^2 + \frac{M}{3}d^2\right)\omega_f$$

$$\omega_f = \frac{-\frac{8d}{3} + 2}{5d/3} = \frac{2/3}{5/6} = 0.8(rad/s)$$

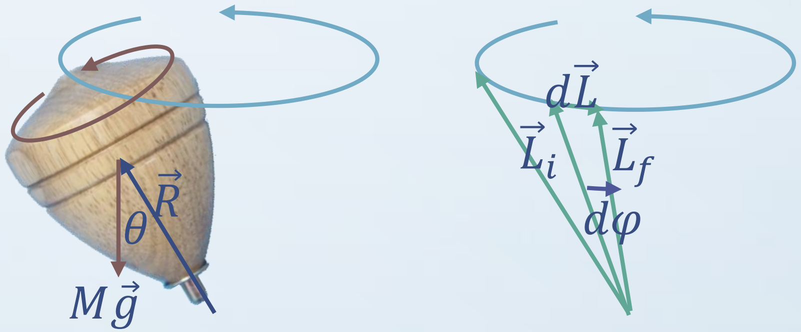


4. THE MOTION OF GYROSCOPES AND TOPS

The Top Motion – Translation, Rotation, Precession



Precession Frequency



$$\vec{\tau} = \vec{R} \times (M\vec{g}), \tau = RMg \sin\theta$$

$$\vec{\tau} = \frac{d}{dt}\vec{L} \qquad d\vec{L} = \vec{\tau}dt \qquad \left|d\vec{L}\right| = RMg \sin\theta \, dt$$

$$d\varphi = \frac{\left|d\vec{L}\right|}{\left|\vec{L}\right| \sin\theta} = \frac{RMg}{L}dt \qquad \omega_p = \frac{d\varphi}{dt} = \frac{RMg}{L} = \frac{RMg}{I\omega}$$

http://www.gifmania.tw/Gifs-Duixiang/Gifs-Toys/Gifs-Spinning-Tops/

4. THE MOTION OF GYROSCOPES AND TOPS

Example: A particle of mass m moves with speed v_0 in a circle of radius r_0 on a frictionless tabletop. The particle is attached to a string that passes through a hole in the table. The string is slowly pulled downward so that the particle moves in a small circle of radius r_1. (a) Find its speed on the circle of radius r_1. (b) Find the tension when the particle is moving in a circle of radius r. (c) Calculate the work done on the particle by the tension.

$$\tau_{net} = 0$$

(a) $$L_i = L_f, r_0 m v_0 = r_1 m v_1, v_1 = \frac{r_0 v_0}{r_1}$$

(b) $$v = \frac{r_0 v_0}{r} \qquad T = m a_r = m\frac{v^2}{r} = \frac{m r_0^2 v_0^2}{r^3}$$

(c) $$W = \int_{r_0}^{r}\left(-\frac{m r_0^2 v_0^2}{r^3}\hat{r}\right)\cdot(dr\hat{r}) = \frac{m r_0^2 v_0^2}{2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)$$

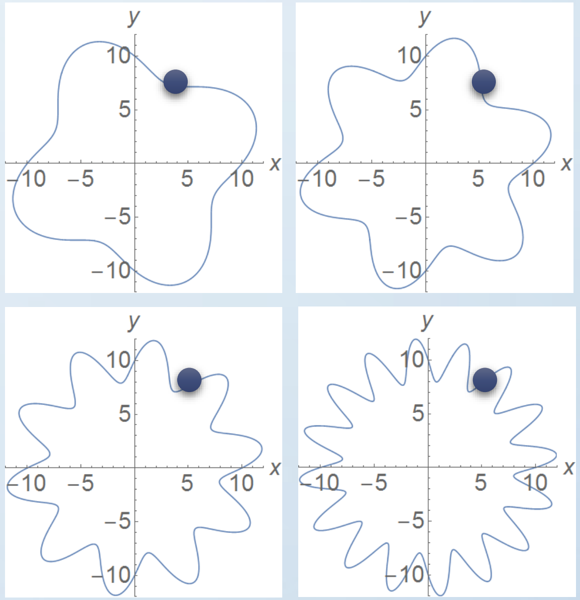
5. QUANTIZATION OF ANGULAR MOMENTUM

The Matter Wave Concept:
All particle motion can be described by wave like $A\sin(kx - \omega t)$.
The orbital motion of a particle strictly comply with the Wilson-Sommerfeld quantization rule:

$$k = \frac{2\pi}{\lambda}, \qquad \oint k ds = n(2\pi)$$

$$k = \frac{p}{\hbar} \qquad \oint \frac{p}{\hbar} ds = n(2\pi)$$

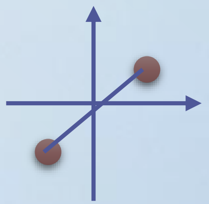
$$2\pi r \frac{p}{\hbar} = 2\pi n \Rightarrow pr = mvr = L = n\hbar$$



It gives the quantization of angular momentum for the electron in the hydrogen atom. (Bohr's model of the hydrogen atom)

5. QUANTIZATION OF ANGULAR MOMENTUM

Example: Consider an oxygen molecule rotating on the xy plane about the z axis. The rotation axis passes through the center of the molecule, perpendicular to its length. The mass of each oxygen atom is 2.66 x 10^{-26} kg, and the average separation between the two atoms is d = 1.21 x 10^{-10} m. Consider the quantization nature, please find the lowest angular speed.



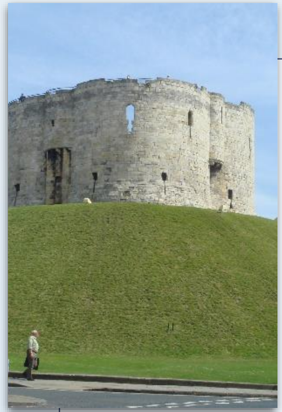
$$I = \sum m_i r_i^2 = 2 \times 2.66 \times 10^{-26} \times \left(0.605 \times 10^{-10}\right)^2$$

$$I = 1.95 \times 10^{-46} (kg\ m^2)$$

$$I\omega = 1 \cdot \hbar, \qquad \omega = \frac{\hbar}{I} = 5.41 \times 10^{11} (rad/s)$$

ACKNOWLEDGEMENT

國立交通大學理學院
自主愛學習計畫



【科技部補助】