Chapter 24 Electric Potential

Physics II – Part I
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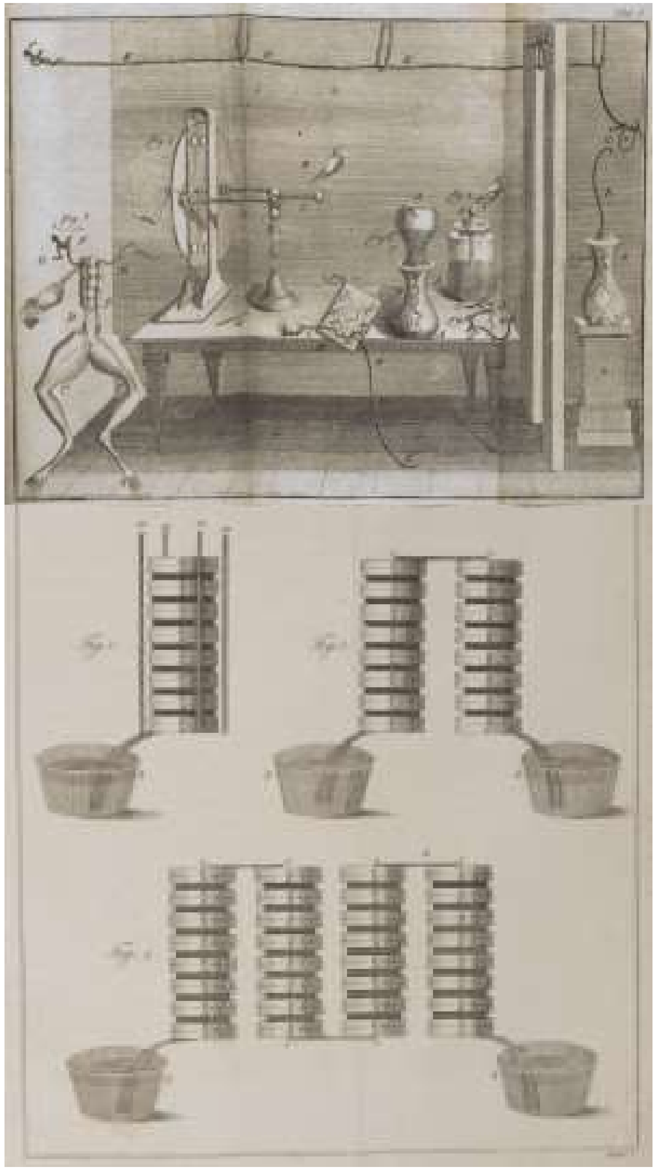
Electrical Cell – Electrochemical Cell

History

1793 AD – Luigi Galvani (Italian)
legs of a frog contract when connected to two different metals

1800 AD – Alessandro Volta (Italian)
voltaic pile, Zn-Cu electrical cell

1800 AD – William Nicholson and Anthony Carlisle (English)
electrolysis, decompose water into O_2 and H_2 gass



Ref: http://libraries.mit.edu/collections/vail-collection/topics/electricity/the-voltaic-pile/

Relation between Charge, Potential Energy, Force, Electric Potential, and Electric Field

Electric Potential

Potential energy and force:

$$dU = -\vec{F} \cdot d\vec{r}, U = -\int \vec{F} \cdot d\vec{r}$$

$$\vec{F} = -\vec{\nabla} U = -\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) U(x, y, z)$$

Electric potential and electric field:

$$\vec{F} = q\vec{E}, U = qV$$

$$dV = -\vec{E} \cdot d\vec{l} \rightarrow V = -\int \vec{E} \cdot d\vec{l}$$

$$\vec{E} = -\vec{\nabla} V(x, y, z)$$

For the 1D case: $V = -\int E dx , E = -\frac{dV(x)}{dx}$

Units of electric potential: 1 Volt = 1 V = 1 J/C

Units of electric field: 1 N/C = 1 V/m

Units of electric potential energy: 1 eV = 1.602×10^{-19} J

Ref:

Uniform Electric Field

Assume $V(x=0)=0$

$$V(x) = -\int_0^x \frac{\sigma}{\varepsilon_0} dx = -\frac{\sigma}{\varepsilon_0} x, x < L$$

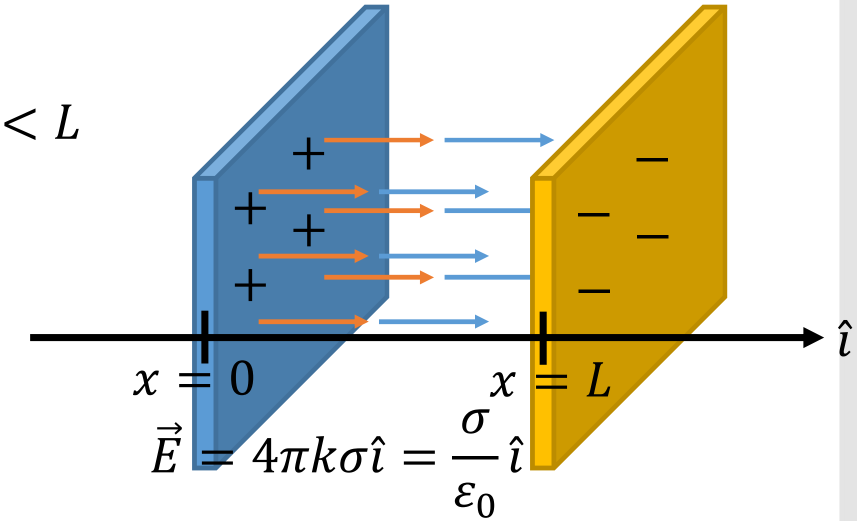
$$V(x) = -\int_0^L \frac{\sigma}{\varepsilon_0} dx - \int_L^x 0 dx$$

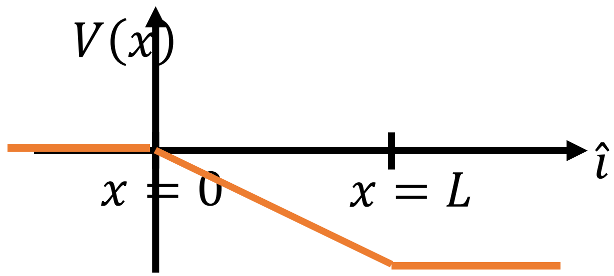
$$V(x) = -\frac{\sigma}{\varepsilon_0} L, x > L$$

$$\Delta V = E_0 L = \frac{\sigma}{\varepsilon_0} L$$

$$V(x) = -\frac{\sigma}{\varepsilon_0} x \rightarrow E = -\frac{dV}{dx} = \frac{\sigma}{\varepsilon_0}$$

The electric potential is continuous.



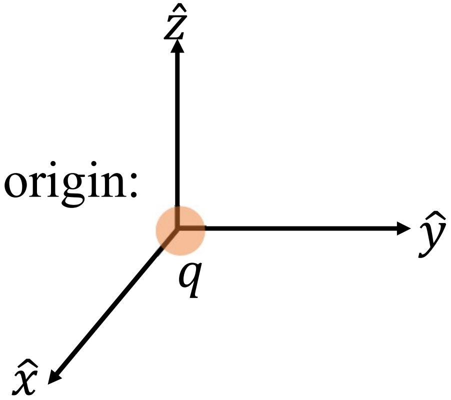


Ref:

Electric Potential of Point Charges

Potential of Discrete Charges

The electric field of a point charge placed at the origin:



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

$$V = -\int \vec{E}\cdot d\vec{r} \qquad d\vec{r} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin(\theta)\,d\phi$$

$$V = -\int_{r_1}^{r_2}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$V = -\int_{\infty}^{r}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr = \frac{q}{4\pi\varepsilon_0 r} = \frac{kq}{r}$$

Ref:

Calculate The Electric Field by Using The Electric Potential

Derive The Electric Field

The relation between the electric field and the electric potential:

$$dV = -\vec{E} \cdot d\vec{r}$$

For the one dimensional electric field and potential, the electric field is calculated as

$$dV(x) = -E(x)dx \rightarrow E(x) = -\frac{dV(x)}{dx}$$

For the two or three dimensional electric potential, the electric field is calculated as

$$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz \qquad \vec{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}$$

$$dV(x, y, z) = -\vec{E} \cdot d\vec{r} = -E_x dx - E_y dy - E_z dz$$

We use the orthogonal coordinate thus the calculation in x is independent of y and z variables.

$$dV = -E_x dx \rightarrow E_x = -\frac{\partial V}{\partial x} \quad \rightarrow E_y = -\frac{\partial V}{\partial y} \& E_z = -\frac{\partial V}{\partial z}$$

Ref:

Calculate Electric Field (Vector Function, Vector Field) from Electric Potential (Scalar Function)

The Gradient Operation

The same relation between $\vec{E}(\vec{r})$ and $V(\vec{r})$:

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}')d\vec{r}' \qquad \vec{E} = -\vec{\nabla}V(\vec{r})$$

With the Cartesian coordinate, we have $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$.

$$\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$$

$$dV = -\vec{E}\cdot d\vec{r} = -E_xdx - E_ydy - E_zdz$$

The three variables are independent: $E_x = -\frac{\partial V}{\partial x}, E_y = -\frac{\partial V}{\partial y}, E_z = -\frac{\partial V}{\partial z}$

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right) = -\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)V$$

$$\vec{E} = -\vec{\nabla}V(\vec{r}) \rightarrow \vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Calculate Electric Field (Vector Function, Vector Field) from Electric Potential (Scalar Function)

The Gradient Operation

The same relation between $\vec{E}(\vec{r})$ and $V(\vec{r})$:

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}')d\vec{r}' \qquad \vec{E} = -\vec{\nabla}V(\vec{r})$$

With the cylindrical coordinate, we have $d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + dz\hat{k}$.

$$\vec{E} = E_r\hat{r} + E_\theta\hat{\theta} + E_z\hat{z}$$

$$dV = -\vec{E}\cdot d\vec{r} = -E_r dr - E_\theta r d\theta - E_z dz$$

The electric field is: $E_r = -\frac{\partial V}{\partial r}, E_\theta = -\frac{\partial V}{r\partial\theta}, E_z = -\frac{\partial V}{\partial z}$.

$$\vec{E} = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{\partial V}{r\partial\theta}\hat{\theta} + \frac{\partial V}{\partial z}\hat{z}\right) = -\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{r\partial\theta} + \hat{z}\frac{\partial}{\partial z}\right)V$$

$$\vec{E} = -\vec{\nabla}V(\vec{r}) \rightarrow \vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{r\partial\theta} + \hat{z}\frac{\partial}{\partial z}$$

Calculate Electric Field (Vector Function, Vector Field) from Electric Potential (Scalar Function)

The Gradient Operation

The same relation between $\vec{E}(\vec{r})$ and $V(\vec{r})$:

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,')d\vec{r}\,' \qquad \vec{E} = -\vec{\nabla}V(\vec{r})$$

With the spherical coordinate, we have $d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}$.

$$\vec{E} = E_r\hat{r} + E_\theta\hat{\theta} + E_\phi\hat{\phi}$$

$$dV = -\vec{E}\cdot d\vec{r} = -E_r dr - E_\theta rd\theta - E_\phi r\sin\theta\, d\phi$$

The electric field is: $E_r = -\frac{\partial V}{\partial r}, E_\theta = -\frac{\partial V}{r\partial\theta}, E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}$.

$$\vec{E} = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{\partial V}{r\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}\right) = -\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{r\partial\theta} + \frac{1}{r\sin\theta}\hat{\phi}\frac{\partial}{\partial\phi}\right)V$$

$$\vec{E} = -\vec{\nabla}V(\vec{r}) \rightarrow \vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{r\partial\theta} + \frac{1}{r\sin\theta}\hat{\phi}\frac{\partial}{\partial\phi}$$

The Ratio of Charge to Mass of Electron

Thomson's measurement of q/m for electrons

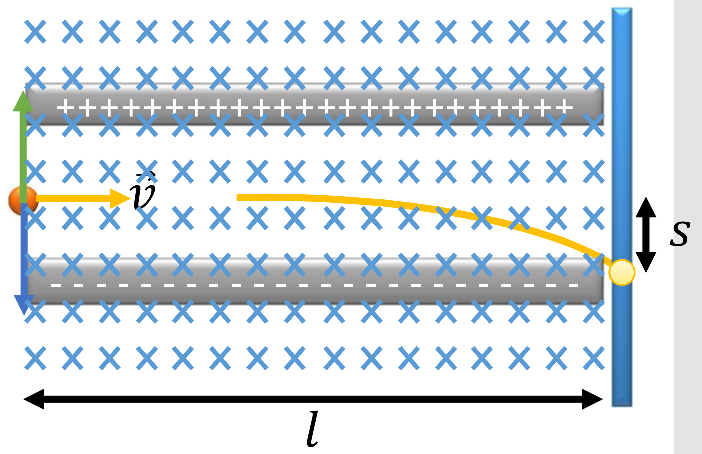
turn on magnetic field to determine $v = E/B$

turn off magnetic field

$$vT = l \rightarrow T = \frac{l}{v} = \frac{Bl}{E}$$

$$s = \frac{1}{2}\frac{qE}{m}T^2 = \frac{qE}{2m}\left(\frac{Bl}{E}\right)^2 = \frac{q}{m}\frac{B^2l^2}{2E}$$

$$\frac{q}{m} = \frac{2Es}{B^2l^2}$$



https://www.nyu.edu/classes/tuckerman/adv.chem/lectures/lecture_3/node1.html

Millikan Oil Drop Experiment

Measurement of The Charge of An Electron

Measure the terminal speed of the uncharged oil drops: $v_{t1} = l_1/t_1$

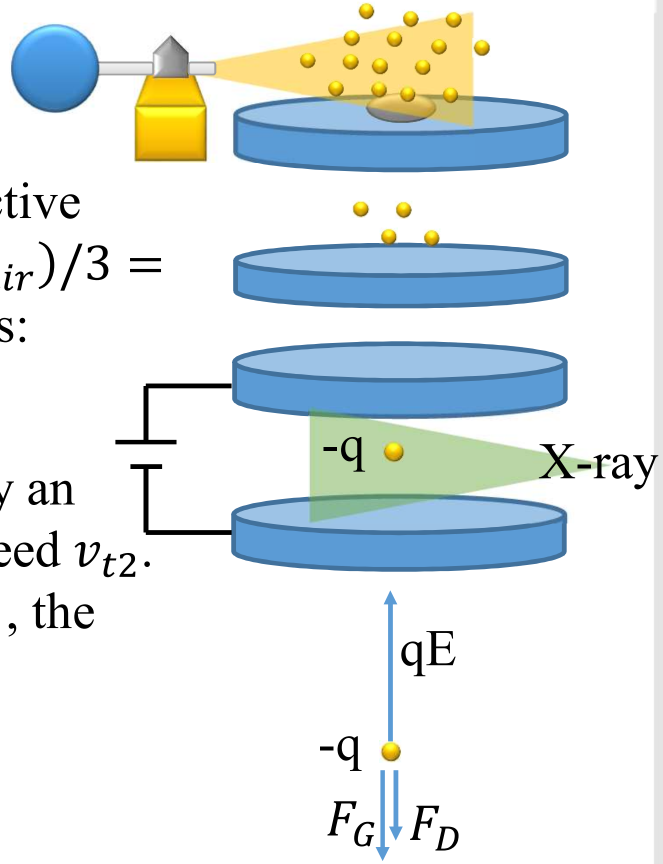
The drag force: $F_D = 6\pi\eta r v_{t1}$, the effective gravitational force: $F_G = 4\pi r^3 g(\varrho - \varrho_{air})/3 = F_D$, the estimated radius of the oil drop is:

$$r = \left(9\eta v_{t1}/2g(\varrho - \varrho_{air})\right)^{1/2}$$

Use X-ray to charge the oil drop & apply an electric field. Determine the terminal speed v_{t2}. The F_G is the same, $F_D = 6\pi\eta r v_{t1} = F_G$, the electric field $qE = qV/d = F_G + F_D' = 6\pi\eta r(v_{t1} + v_{t2})$

$$q = 6\pi\eta r(v_{t1} + v_{t2})d/V$$

From q/m, we can estimate m_e.



http://vlab.amrita.edu/?sub=1&brch=195&sim=357&cnt=1

Potential Difference in a Uniform Electric Field

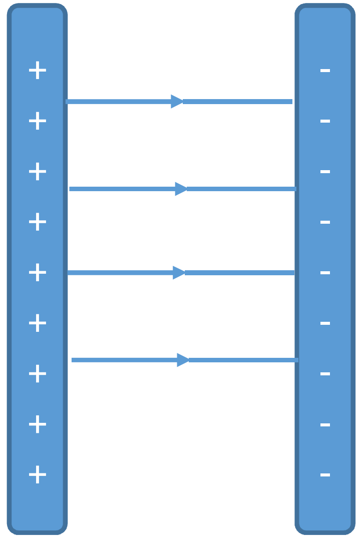
Uniform Electric Field

A proton is in motion in a uniform electric field $E = 8.0 \times 10^4$ V/m in a distance $d = 0.50$ m. (a) Find the change of electric potential. (b) Find the change of electric potential energy.

$$\Delta V = Ed = 4.0 \times 10^4 \text{ V}$$

$$\Delta U = q\Delta V = 1.602 \times 10^{-19} \times 4.0 \times 10^4$$

$$\Delta U = 6.4 \times 10^{-15} \text{ J} = 40 \text{ keV}$$



Ref:

Electric Potential of Point Charges

Potential of Discrete Charges

For a hydrogen atom, please calculate (a) the electric potential at a distance $r = 0.529 \times 10^{-10}$ m away from the proton and (b) the electric potential energy of an electron at this separation distance.

$$V = k\frac{e}{r} = (9 \times 10^9)\frac{1.602 \times 10^{-19}}{0.529 \times 10^{-10}} = 27.2\ \mathrm{V}$$

$$U = -eV = -(1.602 \times 10^{-19}) \times 27.2\ \mathrm{J} = -27.2\ \mathrm{eV}$$

In nuclear fission, a uranium-235 nucleus captures a neutron and splits apart into two lighter nuclei. Assume that the split nuclei of a barium nucleus (charge 56e) and a krypton nucleus (charged 36e) and separated with a distance of 14.6×10^{-15} m, please calculated the potential energy.

$$U = k\frac{q_1 q_2}{r} = (9 \times 10^9)\frac{56 \times 36 \times (1.602 \times 10^{-19})^2}{14.6 \times 10^{-15}} = 199\ Me\mathrm{V}$$

Ref:

Calculate The Electric Field from The Electric Potential

Derive The Electric Field

Fine the electric field for the one-dimensional electric potential of $V(x) = 100 - 25x$ (V).

$$E(x) = -\frac{dV}{dx} = 25\ (\mathrm{V/m})$$

An electric dipole consists of charges $+q$ and $-q$ placed at $a\hat{\imath}$ and $-a\hat{\imath}$, respectively, on the x-axis. Please find the electric potential and electric field at $x > a$ on the x-axis. Please evaluate the electric potential when $x \gg a$. Electric dipole moment is defined as $p = 2qa$.

$$V(x) = k\frac{q}{x-a} + \frac{k(-q)}{x+a} = \frac{2kqa}{x^2 - a^2} = k\frac{p}{x^2 - a^2}$$

$$\vec{E} = -\frac{dV}{dx}\hat{\imath} = \frac{2kpx}{(x^2 - a^2)^2}\hat{\imath}$$

$$x \gg a, V(x) = \frac{kp}{x^2} \rightarrow E = -\frac{dV}{dx} = \frac{2kp}{x^3}$$

Ref:

Calculate The Electric Field Due to a Continuous Charge Distribution

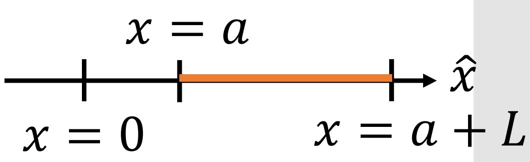
Continuous Charge Distribution

Calculate the electric potential at $x = 0$ for a uniformly charged rod placed from $x = a$ to $x = a + L$ on the x-axis if the total charge on the rod is Q.

$$dq = \lambda dx \ \& \ \lambda = \frac{Q}{L}$$

$$dV = k\frac{dq}{x} = k\frac{Q}{L}\frac{dx}{x}$$

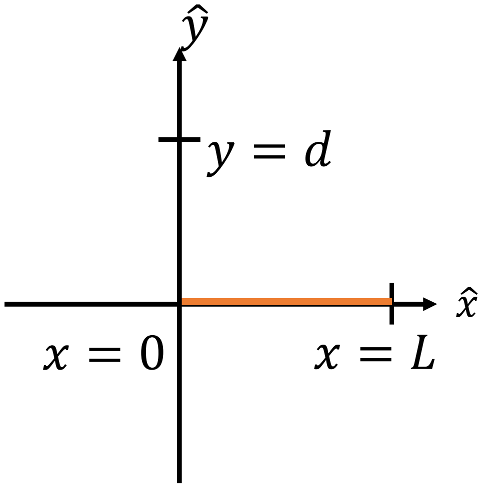
$$V = \frac{kQ}{L}\int_{a}^{a+L}\frac{dx}{x} = \frac{kQ}{L}\left(\ln\left(\frac{a+L}{a}\right)\right)$$



Ref:

Continuous Charge Distribution

Calculate the electric potential at $y = d$ on the y-axis for a uniformly charged rod placed from $x = 0$ to $x = L$ on the x-axis if the total charge on the rod is Q.



$$dq = \lambda dx \;\&\; \lambda = \frac{Q}{L}$$

$$dV = k\frac{dq}{\sqrt{x^2 + d^2}} = k\frac{Q}{L}\frac{dx}{\sqrt{x^2 + d^2}}$$

$$V = \frac{kQ}{L}\int_0^L \frac{dx}{\sqrt{x^2 + d^2}}$$

$$x = d\tan\theta\,,\theta \to (0, \tan^{-1}\left(\frac{L}{d}\right))$$

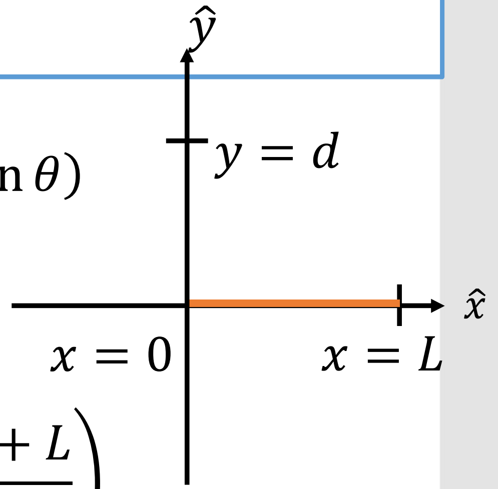
$$V = \frac{kQ}{L}\int_0^{\tan^{-1}(L/d)} \frac{\sec^2\theta\, d\theta}{\sec\theta} = \frac{kQ}{L}\int_0^{\tan^{-1}(L/d)} \frac{\cos\theta\, d\theta}{\cos^2\theta}$$

Ref:

Calculate The Electric Field Due to a Continuous Charge Distribution

Continuous Charge Distribution

Calculate the electric potential at $y = d$ on the y-axis for a uniformly charged rod placed from $x = 0$ to $x = L$ on the x-axis if the total charge on the rod is Q.



$$V = \frac{kQ}{L}\int_0^{\tan^{-1}(L/d)} \frac{1}{2}\left(\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta}\right) d(\sin\theta)$$

$$V = \frac{kQ}{2L}\left[\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right]_0^{\tan^{-1}(L/d)}$$

$$V = \frac{kQ}{2L}\ln\left(\frac{1+L/\sqrt{L^2+d^2}}{1-L/\sqrt{L^2+d^2}}\right) = \frac{kQ}{2L}\ln\left(\frac{\sqrt{L^2+d^2}+L}{\sqrt{L^2+d^2}-L}\right)$$

$$V = \frac{kQ}{2L}\ln\left(\frac{\left(\sqrt{L^2+d^2}+L\right)^2}{d^2}\right) = \frac{kQ}{L}\ln\left(\frac{\sqrt{L^2+d^2}+L}{d}\right)$$

Ref:

Calculate The Electric Field Due to a Continuous Charge Distribution

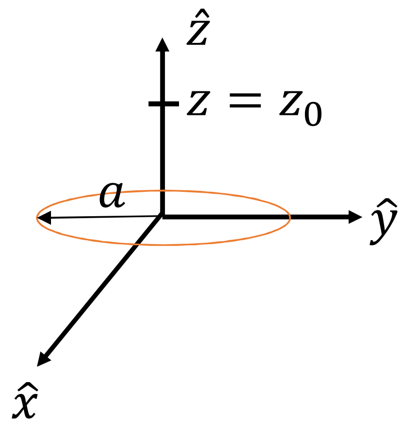
Continuous Charge Distribution

A uniformly charged ring with a radius of a is placed on the xy plane with its central axis aligned with the z-axis. If the total charge on the ring is Q please calculate the electric potential at $z = z_0$ on the z-axis.

$$2\pi a\lambda = Q \rightarrow \lambda = \frac{Q}{2\pi a}$$

$$dV = k\frac{dq}{\sqrt{a^2 + z_0^2}} = \frac{kQ}{2\pi a}\frac{a d\theta}{\sqrt{a^2 + z_0^2}}$$

$$V = \frac{kQ}{2\pi\sqrt{a^2 + z_0^2}}\int_0^{2\pi} d\theta = \frac{kQ}{\sqrt{a^2 + z_0^2}}$$



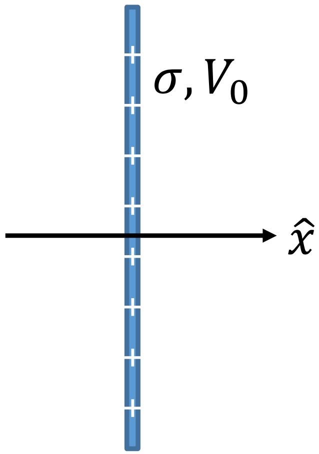
Ref:

Calculate The Electric Potential by Using The Electric Field

Application of Gauss's Law

A uniformly charged plate with charge density of σ is placed on the yz plane at $x = 0$. If the potential on the plate is V_0, please calculate the electric potential as a function of x.

Use Gauss's law to find the electric field:



$$x > 0, E = 2\pi k\sigma\hat{\imath}$$

$$x < 0, E = -2\pi k\sigma\hat{\imath}$$

$$x > 0, V = V_0 - \int_0^x 2\pi k\sigma dx = V_0 - 2\pi k\sigma x$$

$$x < 0, V = V_0 - \int_0^x (-2\pi k\sigma) dx = V_0 + 2\pi k\sigma x$$

$$V = V_0 - 2\pi k\sigma|x|$$

Ref:

Application of Gauss's Law

Calculate The Electric Potential by Using The Electric Field

Please calculate the electric potential of a charged metal shell with a radius of R and a total charge of Q.

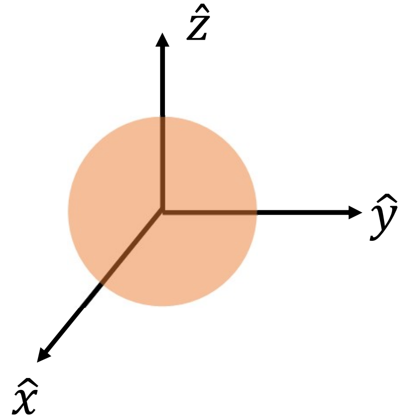
Use Gauss's law to fine the electric field:

$$r > R: E = \frac{kQ}{r^2}$$

$$r < R: E = 0$$

$$r > R, V = -\int_{\infty}^{r} \frac{kQ}{r^2} dr = \frac{kQ}{r}$$

$$r < R, V = -\int_{\infty}^{R} \frac{kQ}{r^2} dr - \int_{R}^{r} 0 dr = \frac{kQ}{R}$$



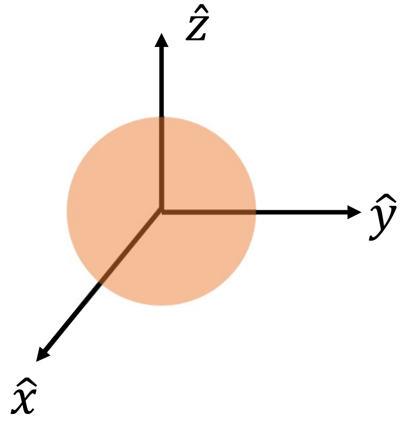
Ref:

Calculate The Electric Potential by Using The Electric Field

Application of Gauss's Law

Please calculate the electric potential of a uniformly charged solid sphere with a radius of R and a total charge of Q.

The volume density is $\rho = 3Q/4\pi R^3$



$$r > R: E = \frac{kQ}{r^2}$$

$$r < R: E = \left(4\pi k\left(\frac{Qr^3}{R^3}\right)\right)/4\pi r^2 = kQr/R^3$$

$$r > R, V = -\int_{\infty}^{r} \frac{kQ}{r^2} dr = \frac{kQ}{r}$$

$$r < R, V = -\int_{\infty}^{R} \frac{kQ}{r^2} dr - \frac{kQ}{R^3}\int_{R}^{r} r dr = \frac{kQ}{R} - \frac{kQ}{R^3}\left(\frac{r^2}{2} - \frac{R^2}{2}\right)$$

$$V = \frac{kQ}{R^3}\left(\frac{3R^2}{2} - \frac{r^2}{2}\right)$$

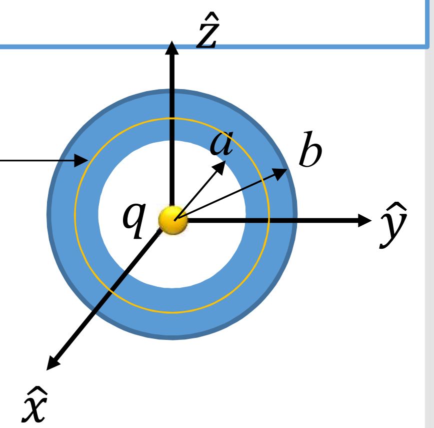
Ref:

Calculate The Electric Potential by Using The Electric Field

Application of Gauss's Law

A hollow uncharged spherical conducting shell has inner and outer radii a and b. A positive charge q is in the cavity and at the center of the sphere. Please find the charge on each surface and find the potential.

The electric field inside the conductor is zero:



$$Q_a = -q, Q_b = q$$

$$r \leq a: E = kq/r^2$$

$$a < r < b: E = 0, r \geq b: E = kq/r^2$$

$$r > b, V = -\int_{\infty}^{r} \frac{kq}{r^2} dr = \frac{kQ}{r}$$

$$b > r > a, V = -\int_{\infty}^{b} \frac{kq}{r^2} dr - \int_{b}^{r} 0 dr = \frac{kQ}{b}$$

$$a > r, V = -\int_{\infty}^{b} \frac{kq}{r^2} dr - \int_{b}^{a} 0 dr - \int_{a}^{r} \frac{kq}{r^2} dr = \frac{kQ}{b} + \frac{kq}{r} - \frac{kq}{a}$$

Ref:

The Equipotential Concept and The Point Discharge Phenomena

Application of Gauss's Law

The two spheres are separated by a distance much greater than R1 and R2. They are connected by a conducting wire. Find the charges Q1 and Q2 on the two spheres if the total charge is Q. Find the ratio of the magnitudes of the electric fields at the surfaces of t

$$Q_1 + Q_2 = Q$$

$$V_1 = V_2 \rightarrow k\frac{Q_1}{R_1} = k\frac{Q_2}{R_2} \rightarrow Q_1 : Q_2 = R_1 : R_2 \rightarrow Q_1 = \frac{R_1}{R_1 + R_2}Q$$

$$Q_2 = \frac{R_2}{R_1 + R_2}Q$$

$$E_1 : E_2 = k\frac{Q_1}{R_1^2} : k\frac{Q_2}{R_2^2} = \frac{R_1}{R_1^2} : \frac{R_2}{R_2^2} = R_2 : R_1$$

Ref:

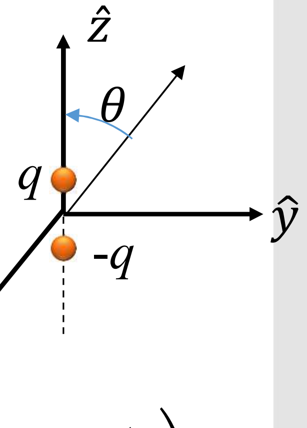
The Potential and Electric Field of an Electric Dipole

Application of Gauss's Law

The two charges of $-q$ and q are placed at $\vec{r}_1 = (0,0,-d)$ and $\vec{r}_2 = (0,0,d)$. Find the electric potential and the electric field of the dipole.

$$V = \frac{kq}{|\vec{r}-\vec{r}_2|} - \frac{kq}{|\vec{r}-\vec{r}_1|}$$

$$V = kq\left(\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{1}{\sqrt{r^2+d^2+2rd\cos\theta}}\right)$$



$$r \gg d \rightarrow V \cong \frac{kq}{r}\left(\left(1-\frac{1}{2}\left(-\frac{2d}{r}\cos\theta\right)\right)-\left(1-\frac{1}{2}\left(\frac{2d}{r}\cos\theta\right)\right)\right)$$

$$V = \frac{kq}{r}\frac{2d\cos\theta}{r}, p = 2qd \rightarrow V = \frac{kp\cos\theta}{r^2}$$

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} - \frac{\partial V}{r\partial\theta}\hat{\theta}$$

Ref: