# Chapter 29 Electric Currents and Magnetic Fields

Physics II – Part II

Wen-Bin Jian
Department of Electrophysics, National Chiao Tung University

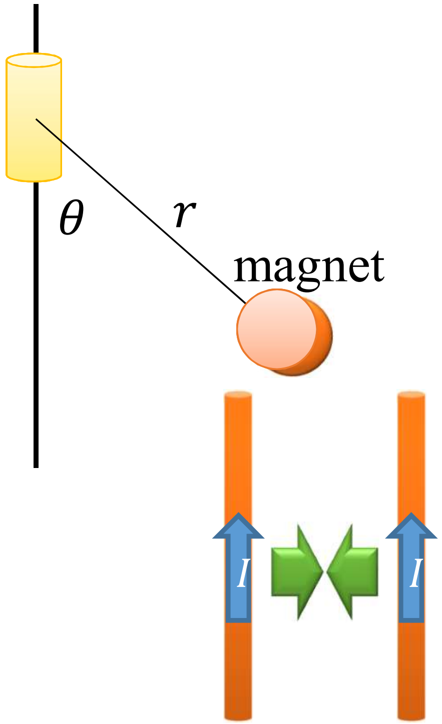
Biot & Savart's Experiments

# History

1820 AD – Hans Christian Ørsted (Danish)
Electric current generates magnetic fields.

1820 AD (18 December) – Jean-Baptiste Biot (French) & Félix Savart (French)
Experimental determination of current generated magnetic fields. Treat currents as magnets with magnetic fields.

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \hat{r}}{r^2} \qquad \frac{\mu_0}{4\pi} = 10^{-7}\ T\ m\ /A$$



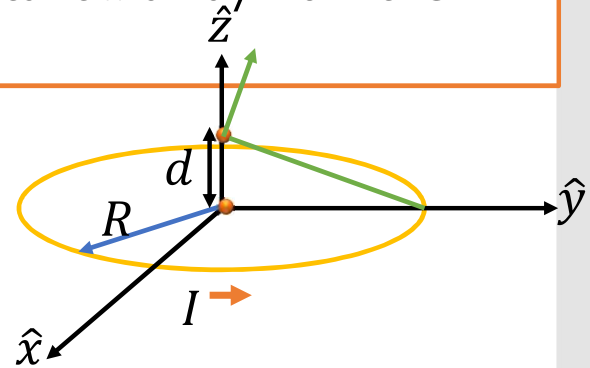

1820 AD (4 December) – André-Marie Ampère (French)
interaction between currents

1882 AD – William Thomson, Baron Kelvin (Scottish)
current balance design

Ref: http://www.ampere.cnrs.fr/parcourspedagogique/zoom/courant/formule/index-en.php

# Magnetic Fields

Use the Biot-Savart law to calculate the magnetic field at the center of a circular loop with radius $R$ and current $I$. Please calculate the magnetic field on the central axis and at a distant $d$ away from the center.




$$\vec{B}_o = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times (-\hat{r})}{r^2} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R d\theta}{R^2} \hat{k}$$

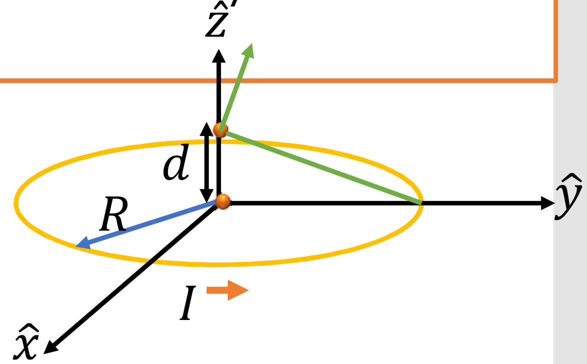
$$\vec{B}_o = \frac{\mu_0 I}{4\pi} \frac{2\pi}{R} \hat{k} = \frac{\mu_0 I}{2R} \hat{k}$$

---

$$\vec{B}_d = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times (-\hat{r})}{r^2} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R d\theta}{R^2 + d^2} \frac{R}{\sqrt{R^2 + d^2}} \hat{k}$$

# Magnetic Fields

Use the Biot-Savart law to calculate the magnetic field at the center of a circular loop with radius $R$ and current $I$. Please calculate the magnetic field on the central axis and at a distant $d$ away from the center.




$$\vec{B}_d = \frac{\mu_0 I}{4\pi} \frac{R^2 2\pi}{(R^2 + d^2)^{\frac{3}{2}}} \hat{k} = \frac{\mu_0 I R^2}{2(R^2 + d^2)^{\frac{3}{2}}} \hat{k}$$

$$B_d \xrightarrow[d \gg R]{} \frac{\mu_0 I R^2}{2d^3} = \frac{2\mu_0 \pi I R^2}{4\pi d^3} = \frac{\mu_0}{4\pi} \frac{2m}{d^3} \Leftrightarrow E = \frac{1}{4\pi\varepsilon_0} \frac{2p}{r^3}$$

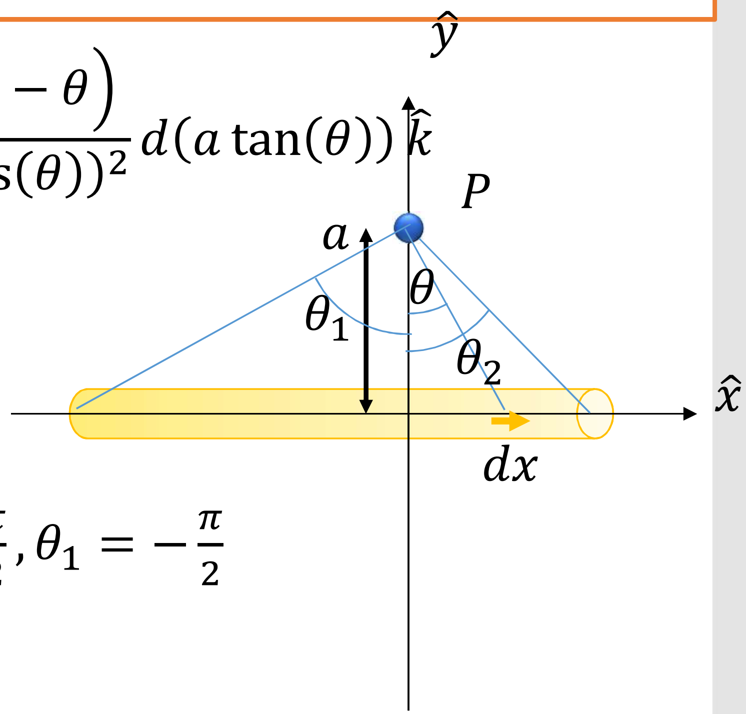
# Magnetic Fields

Use the Biot-Savart law to calculate the magnetic field at point P a distance $a$ away from the segment of line current $I$.

$$\vec{B}_P = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2} \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{(a/\cos(\theta))^2} d(a\tan(\theta))\,\hat{k}$$

$$\vec{B}_P = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2} \frac{\cos(\theta)\,a}{a^2} d\theta\,\hat{k}$$

$$\vec{B}_P = \frac{\mu_0 I}{4\pi a}(\sin(\theta_2) - \sin(\theta_1))\hat{k}$$




An infinitely long line current, $\theta_2 = \frac{\pi}{2}, \theta_1 = -\frac{\pi}{2}$

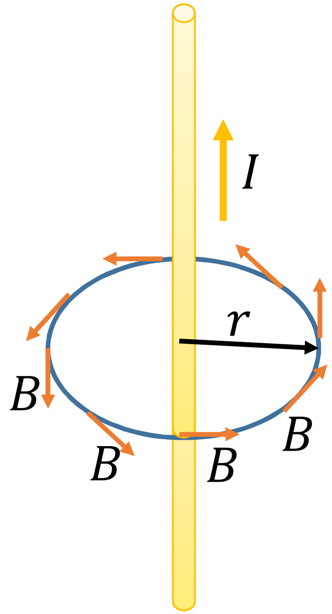
$$\vec{B}_P = \frac{\mu_0 I}{2\pi a}\hat{k}$$

# Ampere's Law & Applications for The Calculation of Magnetic Fields

## Ampere's Law

Magnetic fields make a circular rotation around the current lead. The magnitude of the magnetic field is of the same value of $\mu_0 I/2\pi r$. Line integral of the field gives

$$\oint \vec{B} \cdot d\vec{l} = \int_0^{2\pi} \frac{\mu_0 I}{2\pi r} r d\theta = \mu_0 I$$




Ampere's Law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

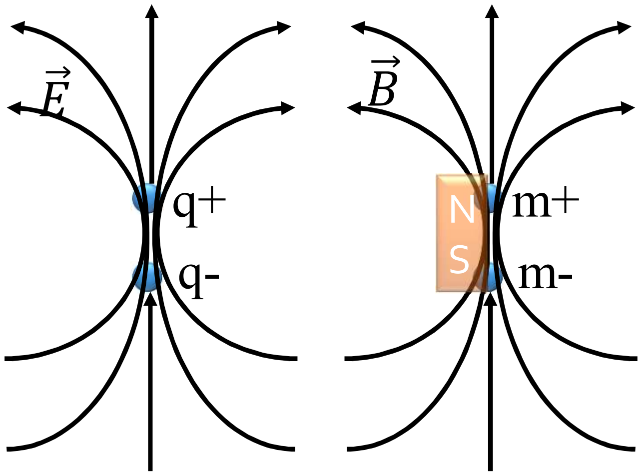
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

There Are No Magnetic Monopoles.

# Gauss's Law for Magnetic Fields

Magnetic moment (magnetization) possesses the same concept as magnetic dipoles.

In analogy to electric dipole, the Gauss's law gives us zero magnetic charges in a box due to the same number of $m+$ and $m-$.




$$\oiint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0}\big(+Nq + (-Nq)\big) = 0$$

$$\oiint \vec{B} \cdot d\vec{A} = \mu_0\big(+Nm + (-Nm)\big) = 0$$

Ref: https://phys.org/news/2016-08-mysterious-magnetic-monopole.html

# Magnetic Force

## Magnetic Force Between Two Parallel, Current-Carrying Wires

Use Ampere's law to find the magnetic field around an infinite longe wire carrying current $I$. Use Lorentz law to find the force per unit length between two parallel current-carrying and straight wires.

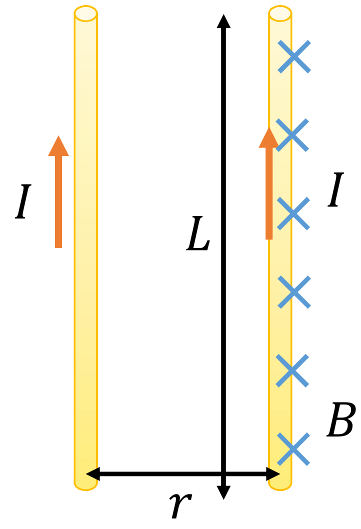
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \rightarrow 2\pi r B = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

$$F = ILB = \frac{\mu_0 I^2}{2\pi r} L \quad \Rightarrow \quad \frac{F}{L} = \frac{\mu_0 I^2}{2\pi r}$$

If $r = 1$ m, $I = 1$ A,

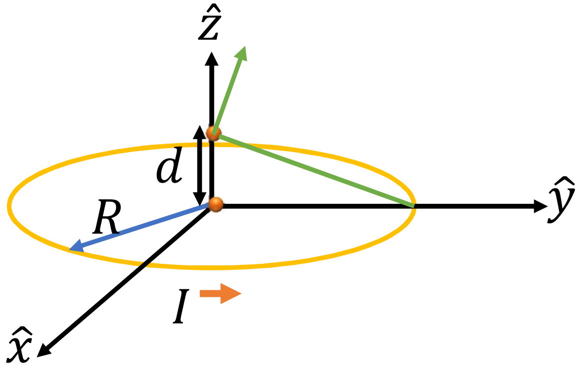
$$\frac{F}{L} = 2(10^{-7}) \frac{1^2}{1} = 2 \times 10^{-7}\ N/m$$



The Magnetic Field of a Solenoid

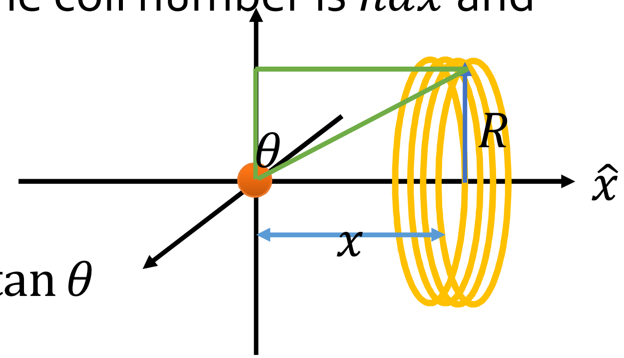
# Solenoid

$$\vec{B}_d = \frac{\mu_0 I R^2}{2(R^2+d^2)^{\frac{3}{2}}}\hat{k}$$




Let the current be $I$ and the number of coils per unit length $n$ for the solenoid. With a small displacement $dx$, the coil number is $ndx$ and the current is $Indx$.

$$d\vec{B} = \hat{\imath}\frac{\mu_0 R^2}{2(R^2+x^2)^{\frac{3}{2}}}Indx$$




$$B = \frac{\mu_0 n I R^2}{2}\int_{x_1}^{x_2}\frac{dx}{(R^2+x^2)^{\frac{3}{2}}}$$

Let $x = R\tan\theta$

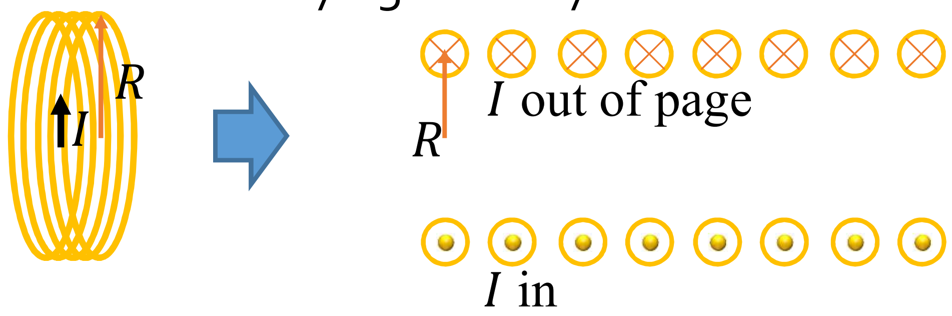
$$B = \frac{\mu_0 n I R^2}{2}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{1}{2}\mu_0 n I(\sin\theta_2 - \sin\theta_1)$$

For an infinite long solenoid, $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$, we get $B = \mu_0 n I$

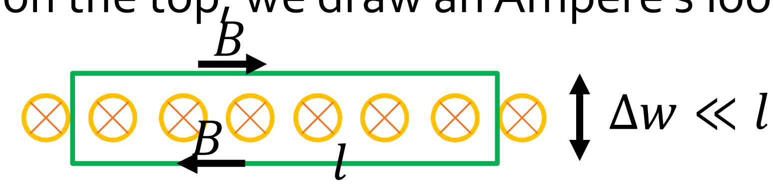
# Solenoid

## The Magnetic Field of a Solenoid

Using the Ampere's law to find the magnetic field of a very long solenoid. Consider $n$ closely wound turns per unit length on a cylinder of radius $R$ and carrying a steady current $I$. Turn to see the crosssection.




For the wires of current on the top, we draw an Ampere's loop and find the magnetic field by




$$Bl + Bl = \mu_0 n l I$$

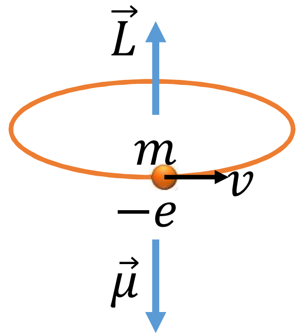
$$B = \frac{1}{2}\mu_0 n I$$

Add up the magnetic field due to the wires of current on the bottom:

$$B = \mu_0 n I$$

# Magnetism in Atoms

Estimate the orbital magnetic moment of the electron in an hydrogen atom.




$\vec{L} = \vec{r} \times (m\vec{v}) \rightarrow L = mvr$, the quantization rule gives you $L = n\hbar$

Estimate the current due to the charge $-e$.

$$I = -\frac{e}{T} = -\frac{e}{\frac{2\pi r}{v}} = -\frac{ev}{2\pi r}$$

Estimate the magnetic moment.

$$\mu = IA \rightarrow \mu = -\frac{ev}{2\pi r}\pi r^2 = -\frac{evr}{2} = -\frac{e}{2m}mvr = -\frac{e}{2m}L$$

Consider the quantization effect, you have $\mu = -n\frac{e\hbar}{2m}$. Here we define the Bohr magneton $\mu_B = \frac{e\hbar}{2m}$ as a unit for atomic magnetic moments.

$$\mu = -n\mu_B$$

# Magnetism in Atoms

Similar to the quantization of orbital angular momentum as:

$$\mu_L = -\frac{e}{2m}L = -n\frac{e\hbar}{2m} = -\frac{\mu_B}{\hbar}n\hbar = -g_L\frac{\mu_B}{\hbar}L$$

Considering the relativistic effect, the Dirac equation results in another quantization, spin quantization, giving another similar equation:

$$\mu_S = -g_S\frac{\mu_B}{\hbar}S$$

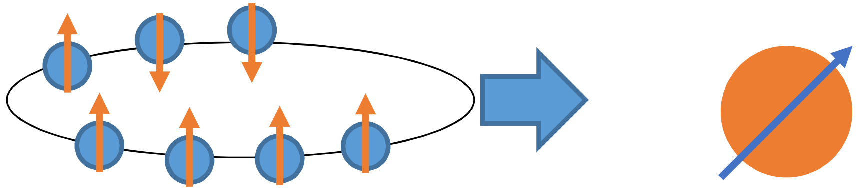
The differences are listed below.

$$g_L = 1, g_S = 2, L = n\hbar, S = \frac{n}{2}\hbar$$

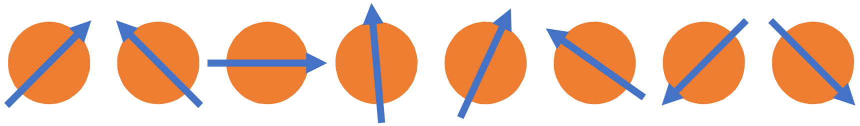
| | Fe (BCC) | Co (HCP) | Ni (FCC) |
|---|---|---|---|
| Bohr Magnettons per Atom | 2.22 | 1.72 | 0.60 |

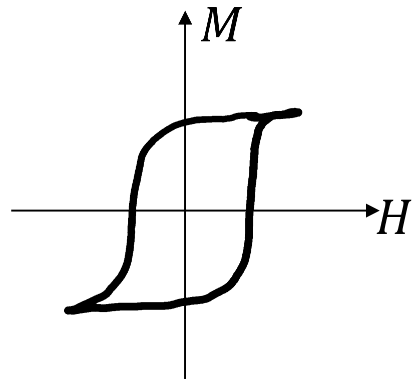
# Magnetism in Matter

Unfilled orbitals, especially *d*-orbital electrons, give atomic magnetic moment:

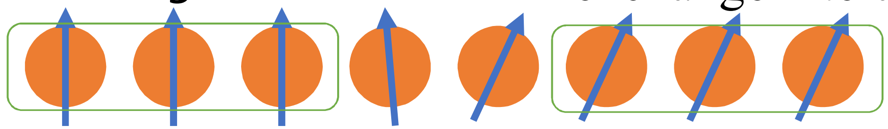


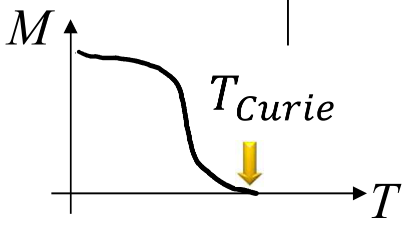
Paramagnetic materials:



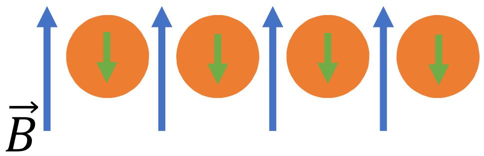



Ferromagnetic materials: exchange interactions






Diamagnetic materials – Lenz' rule:

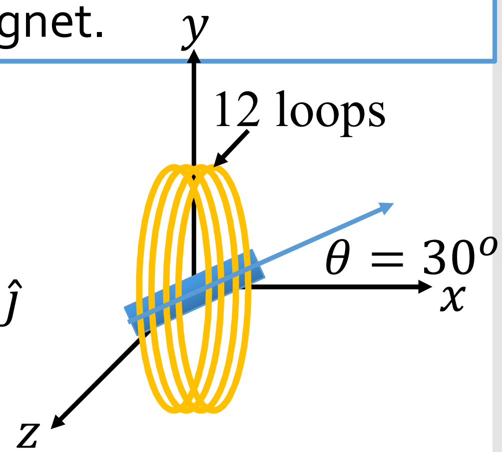


# Examples

A magnet of magnetic moment $m$ is placed at the origin on the $xz$ plane with a direction making an angle of $\theta = 30^o$ to the $x$-axis. A coil of 12 loops is fixed at the origin with its central axis along the $x$-axis. Please calculate the magnetic torque on the magnet.

$$\vec{B} = 12\frac{\mu_0 I}{2R}\hat{\imath}$$

$$\vec{\tau} = \vec{m} \times \vec{B} = -\frac{6m\mu_0 I}{R}\sin(30^o)\,\hat{\jmath} = -\frac{3m\mu_0 I}{R}\hat{\jmath}$$



Calculation of The Magnetic Field

# Examples

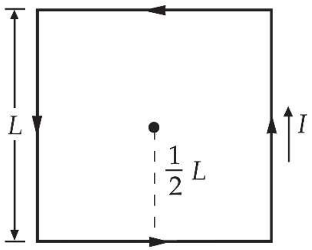
Find the magnetic field at the center of a square loop of edge length $L$, which carries a current $I$.

Use the previously derived equation with $\theta_1 = -\pi/4$ and $\theta_2 = \pi/4$ for one edge of the square loop.

$$B = \frac{\mu_0 I}{4\pi\, r}(\sin(\theta_2) - \sin(\theta_1))$$

$$B_{one\ edge} = \frac{\mu_0 I}{4\pi(L/2)}(\sin(\pi/4) - \sin(-\pi/4)) = \frac{\mu_0 I}{2\pi L}\sqrt{2}$$

$$B_{total} = 4\frac{\mu_0 I}{2\pi L}\sqrt{2} = \frac{2\mu_0 I}{\pi L}\sqrt{2}$$



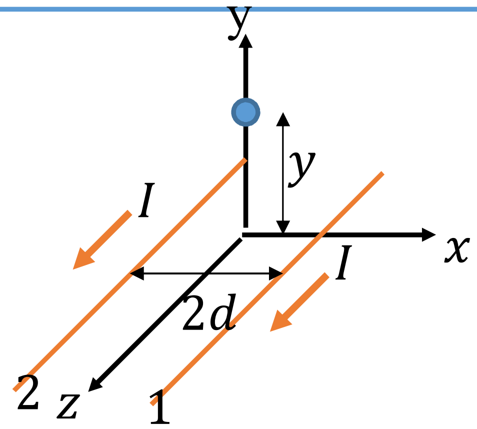
# Examples

Two infinite wires carrying current $I$ are placed on the $xz$ plane with their central axes parallel to the $z$-axis. They are symmetrically arranged on both sides of the $z$-axis with a separation of $2d$. Please calculate the magnetic field at the position of $(0, y, 0)$.

$$B_{wire} = \frac{\mu_0 I}{2\pi r} \qquad r = \sqrt{d^2 + y^2}$$

$$\vec{B}_1 = \frac{\mu_0 I}{2\pi\sqrt{d^2 + y^2}}\left(-\frac{\sqrt{2}}{2}\hat{\imath} - \frac{\sqrt{2}}{2}\hat{\jmath}\right)$$

$$\vec{B}_{total} = \frac{\mu_0 I}{2\pi\sqrt{d^2 + y^2}}\left(-\sqrt{2}\hat{\imath}\right)$$



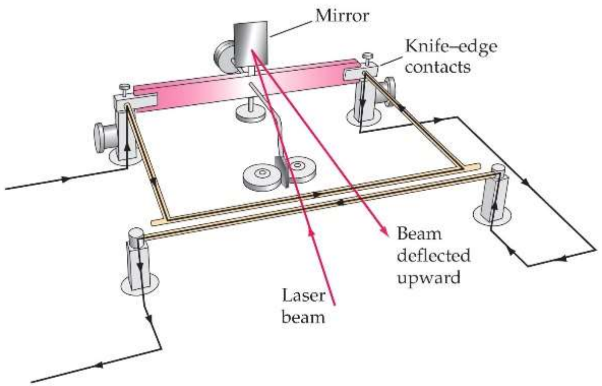
The Lorentz Force

# Examples

Two straight rods of $L$ in length are separated at a distance $d$ apart in a current balance. The carry currents are $I$ each in opposite direction. What mass must be placed on the upper rod?

The magnetic field due to an infinite wire is

$$B = \frac{\mu_0 I}{2\pi d}$$




The repulsive Lorentz force is

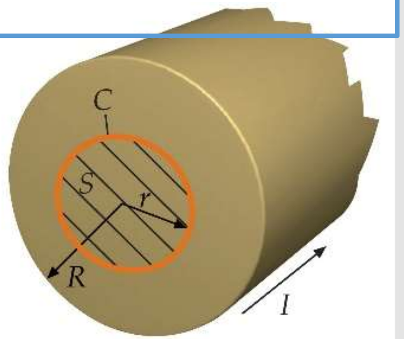
$$F = IlB = IL\frac{\mu_0 I}{2\pi d}$$

$$mg = \frac{\mu_0 I^2 L}{2\pi d} \to m = \frac{\mu_0 I^2 L}{2\pi g d}$$

# Ampere's Law & Applications for The Calculation of Magnetic Fields

## Examples

A long, straight wire of radius *R* carries a current *I* that is uniformly distributed over the circular cross section of the wire. Find the magnetic field both outside and inside the wire.




Current & Current Density:

$$I = JA \rightarrow J = \frac{I}{A} = \frac{I}{\pi R^2}$$

$$r > R: \int \vec{B} \cdot d\vec{l} = B 2\pi r = \mu_0 I \rightarrow B = \frac{\mu_0 I}{2\pi r}$$

$$r < R: \int \vec{B} \cdot d\vec{l} = \mu_0 I \rightarrow 2\pi r B = \mu_0 (J \pi r^2)$$

$$B = \frac{\mu_0 J r}{2} = \frac{\mu_0 I r}{2\pi R^2}$$

# Examples

A wire carrying a current $I$ is bent to form a toroid with $N$ loops. Please find the magnetic field inside and outside the toroid.

Outside the toroid:

$$r > b: 2\pi r B = \mu_0 \times 0 \rightarrow B = 0$$

$$r < a: 2\pi r B = \mu_0 \times 0 \rightarrow B = 0$$

Inside the toroid:

$$b > r > a: 2\pi r B = \mu_0 \times NI \rightarrow B = \frac{\mu_0 NI}{2\pi r}$$